Discovering Early de Finetti's Writings on Trivalent Theory of Conditionals

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Abstract

The trivalent and functional theory of the truth of conditionals developed by Bruno de Finetti has recently gathered renewed interests, particularly from philosophical logic, psychology and linguistics. It is generally accepted that de Finetti introduced his theory in 1935. However, a reading of his first publications indicates an earlier conception of almost all his theory. We bring to light a manuscript and unknown writings, dating back to 1928 and 1932, detailing de Finetti's theory. The two concepts of thesis and hypothesis are presented as a cornerstone on which logical connectives are established in a 2-to-3 valued logic. The proposed generalisation of the bivalent material implication to the trivalent framework, based on the bivalent entailment is however different from the one that will be introduced in 1935. In these early writings de Finetti presents original results that will later be independently rediscovered by other researchers. In particular, the 'suppositional logic' developed by Theodore Hailperin in 1996 presents numerous similarities. Conversely, we consider the notion of validity proposed by Hailperin in line with de Finetti's approach. Overall we attribute the primacy of the trivalent theory to de Finetti; this early conception enabled him to take an original position and argue with Hans Reichenbach.

Keywords: De Finetti's pioneer contributions to conditionals, de Finetti's conditional, Trivalent semantics, 2-to-3 valued logic validity.

1. Introduction: De Finetti Formed Its Trivalent Theory of Conditionals before 1935

For 20 years the trivalent theory of the truth of conditionals, proposed by Bruno de Finetti (1906-1985), has gathered numerous interests in various fields such as Philosophical Logic (e.g. Milne 1997; Mura 2009; Vidal 2014; Égré, Rossi and Sprenger 2020a, 2020b), Linguistics (e.g. Rothschild 2014; Douven 2016; Lassiter 2020; Lassiter and Baratgin 2021), Artificial Intelligence (e.g. Dubois and Prade 1994; Coletti and Scozzafava 2002), Psychology (e.g. Baratgin, Over and Politzer

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2013, 2014; Baratgin and Politzer 2016; Baratgin, Politzer, Over et al. 2018; Nakamura, Shao, Baratgin et al. 2018; Politzer, Jamet and Baratgin 2020) and Didactics (e.g. Delli Rocili and Maturo 2013).

According to de Finetti (1936), the indicative conditional *If* E_2*,* E_1 is a tri-event which is true if E_2 and E_1 are true, is false if E_2 is true and E_1 is false, and is otherwise undefined ('null' or 'void' truth value). More importantly, the indicative conditional is always undefined when its antecedent is false. For de Finetti, a tri-event can be understood through an analogy with a conditional *bet on If* E_2*,* E_1. This bet is won when E_2, E_1 are realized, lost when E_2 is realized but not E_1 and called off when E_2 is not realized. De Finetti proposes a trivalent logic system superimposed on traditional bi-valued logic with in addition to conditional, the usual connectives of negation, conjunction, disjunction and (material) implication.

The notion of 'tri-event' is the first essential step in de Finetti's approach, to define conditional probability as the result of a coherent subjective assessment. In his famous lecture to *Institut Henri Poincaré* in Paris, de Finetti (1937: 13-14) first demonstrates that the subjective coherent evaluation of probabilities by a given individual of an event always ranges between 0 and 1 and that the sum of the assessed probabilities of incompatible events makes 1. De Finetti uses a demonstrative method that entails: (i) to define an analogical unconditional bet on events E_i; (ii) to write the linear equation system of gains as a function of stakes S_i and outlaid pays $-P_iS_i$ with P_i probabilities of E_i evaluated by a given individual; and (iii) to apply the coherence constraint (not to lose the bet for sure) on this linear equation system. He then generalises this method to define the conditional probability with the following four successive steps:

i. The tri-event *if* E_2 *then* E_1 is presented through the analogy with a conditional bet in the particular situation where E_1 implies E_2. The bet is won when E_1 (and therefore E_2), lost when E_2 and not E_1 and called off when not E_2.

ii. The expressions of three possible gains (G, G_1, G_2) for the three possible outcomes in function of their stakes (S, S_1 and S_2) and the outlaid pays (pS, p_1S_1, p_2S_2) give a linear equation system of three equations:

$$\text{(CP)} \begin{cases} G \ = (1-p)S + (1-p_1)S_1 + (1-p_2)S_2 \\ G_1 = -pS - p_1S_1 + (1-p_2)S_2 \\ G_2 = 0 - p_1S_1 - p_2S_2 \end{cases}$$

iii. The notion of coherence (not to lose the bet for sure) gives a constraint on the linear equation system CP (its determinant must be null otherwise the stakes can be set so that the gains have arbitrary values, possibly all positive) requiring the relation $p_1 = pp_2$.

iv. Considering the general case $E_1 \cap E_2$ (and not simply E_1 with E_1 implies E_2) de Finetti obtains the conditional probability $P\left(\frac{E_1}{E_2}\right) = \frac{P(E_2 \cap E_1)}{P(E_2)}$ from which the Bayes' rule is derived.[1]

It is commonly accepted that de Finetti introduced his tri-event theory in 1935 (de Finetti 1936), at the Sessions on *Induction and Probability* of the *First Congress for the Unity of Science* (International Congress of Scientific Philosophy) in Paris (see Galavotti 2018: for an in-depth analysis of these sessions). Later writings, make reference in a rather scattered way to his trivalent logic. De Finetti refers to it through an analogy with the bet schema to define the conditional event to illustrate the logic of uncertainty underlying probabilities (de Finetti 1980: 1164-1165) or again to discuss the quantum logic (see appendix of de Finetti [1970] 1975: 304-313).

However as underlined by Mura (2009), the idea of a third truth value as 'null' when the antecedent is false was already present in 1934 in the book *L'invenzione della verità* published posthumously in 2006 (see de Finetti [1934] 2006: 103). Hence, one may wonder when did de Finetti really conceive his theory? Evidence indicates it happened before 1931.

Indeed, from 1930s onward, numerous arguments in favour of a subjective logic more general than the traditional objective logic can be found in several early publications of de Finetti (see de Finetti 1930; de Finetti 1931; de Finetti 1933; de Finetti [1934] 2006). The elements described then would support the logic of probabilities, the exposure of the betting scheme in an unconditional framework and also the presentation of the notion of coherence. Notably in the Memoria, dated 'Rome june 4 1930', *Sul significato soggettivo della probabilità*, de Finetti (1931) already exposes the demonstrative method in an unconditional situation that will be generalized to the conditional bet in de Finetti (1937). Yet, de Finetti (1931) foresees in his conclusion that the same process may allow the definition of conditional probability:

> It will then be observed that no mention has ever been made here of subordinate probabilities (probability that an event occurs when another event is supposed to occur),

[1] For the sake of consistency with the rest of the document we use from now on the same notations and terminology used in the original de Finetti's manuscripts presented and discussed in section 2. Thus, we use: "$-$" for "negation" or "opposite", "$\cap$" and "$\cup$" for respectively "product" and "sum" (As noted by Mura (2009: 203), de Finetti 1936 uses neither the logic terms of "conjunction" nor "disjunction". De Finetti's goal is to construct a ternary logic that supports the subjective probability theory. Consequently he uses the same vocabulary as used in probability theory), ":=:" for equivalence and "=" defined as $a = b :=: a^{\cup} b \supset a_{\cap} b$. Tri-event are noted "$\frac{E_1}{E_2}$" instead of the modern notation "$E_1 \mid E_2$". We remain faithful as much as possible to the term "subordinate" traditionally used in Italian and French mathematical papers at that time rather than to use the modern term "conditional". These two terms are considered similar in the literature (e.g. de Finetti 1967). A subordinate conditional clause is used when a fact or action is necessary before another fact or action is carried out. The subordinate clause is more general. It conveys two kinds of information: foreground information (i.e. open information), the communication of which is the subordinate's actual task, and background information (i.e. implied, epiphenomenal, incidental information concerning the subject's opinion of the speaker) generally known as a "presupposition" (Ducrot 1969). De Finetti's tri-event can account for the "elementary presupposition" (e.g. Beaver and Krahmer 2001; Ducrot [1980] 2008). De Finetti seemed to agree with this idea (see Mura's notes 9 and 12, 174-175 in de Finetti [1979] 2008).

of the relative theorem of compound probabilities, and of the resulting concept of independent events. These notions are much more delicate and cannot be ordinarily judged, and that it is not at all necessary to introduce them to start with. One can, and indeed it is advisable, if one wants to make the concepts clear, first develop the theory of the probabilities of an event, a theory of which we have given all the foundations here, and then leave to a second time the extension of the calculation of probabilities to subordinate events, an extension that needs support, definitions and explanations that are completely new and conceptually interesting. This topic will also be the subject of other work. We observe, however, from now on, that, if we want to be satisfied with a definition without psychological content, as usually given, we would already have all the elements to define 'formally' the subordinate probability, calling 'probability of E_1 subordinate to E_2' = $\frac{P(E_1 \cap E_2)}{P(E_2)}$. From it, would immediately result, the theorem of the compound probabilities:

$$P(E_1 \cap E_2) = P(E_2) \times P\left(\frac{E_1}{E_2}\right),$$

if we indicate $P\left(\frac{E_1}{E_2}\right)$ the probability of E_1 subordinated to E_2; such theorem, however, would only be a concealed definition of the symbol $P\left(\frac{E_1}{E_2}\right)$. In the way of proceeding that we will develop and have announced here, we will instead give a direct psychological definition of subordinate probabilities thanks to which the theorem on compound probabilities (and therefore the 'formal' definition indicated here) results as a necessary consequence of the usual definition of coherence. And this is the only way of proceeding in accordance with our point of view (de Finetti 1931: 328-329, our translation).[2]

Therefore de Finetti has certainly developed, as early as 1930, the concept of tri-event confirming once again the assertion of Morini[3]

> that de Finetti's theory takes an almost definitive form since the very beginning of his research. It was between 1929 and 1931 that his theory of probability took shape, which he continued to defend throughout his rich scientific career. The almost 300 articles he wrote, the first of which were mostly mathematical in content, are a re-elaboration and deepening of the ideas he had conceived at the age of twenty (Morini 2007: 3-4, our translation).

De Finetti's original writings were acquired by the University of Pittsburgh and stored in the Archives of Scientific Philosophy, alongside other writings representing the so-called 'philosophy of science' of the last century (Ramsey, Carnap, Reichenbach, Hempel, Feigl and Salmon). Among these documents, two folders containing original writings on tri-events are of interest to us.

- *Box 6, Folder 2* entitled *"Logica plurivalente", 1927-1935* (see figure 1) and cited here as de Finetti 1927-1935. In addition to handwritten and typed drafts of de Finetti's (1936) presentation, it contains a correspondence with Hans Reichenbach (1891-1953) dated 1935 as well as original writings, text

[2] De Finetti (1931) uses E and E' instead of E_1 and E_2 and '.' instead of '$\cap$'.
[3] For example, de Finetti's concept of *random exchangeable sequences* dates back to 1930 (Bassetti and Regazzini 2008), as well as his criticism on countable additivity (Regazzini 2013).

and mixed drafts dated 1927.[4] Notably, there are 7 pages of typewritten text dated "Rome, Sunday September 16 1928" and titled: *l'EVENTO SUBORDINATO* [5] *come ente logico* [The subordinate event as a logical entity] de Finetti 1927-1935: 154-60, #'BD-06-02-55' and cited here as de Finetti 1928a, one manuscript page dated 'Rome, March 18 1928' titled *Logica degli eventi* [Logic of events] (de Finetti 1927-1935: 173, #'BD-06-02-66') and cited here as de Finetti 1928c and several draft sheets which were used to establish the demonstrations of the writings.[6]

- *Box 5, Folder 10* entitled *Lezioni sulla probabilità* [Lessons on probability], dated 1932-1933 and cited here as de Finetti 1932a. This folder corresponds to the manuscript of lectures that de Finetti gave in 1932-1933 at Trieste University.[7] The notion of tri-event is synthetically literally presented (26-28).

We will analyse how de Finetti's early approach and methods differ from the presentation of de Finetti 1936. We will underline the original results that were later rediscovered independently by other authors such as Theodore Hailperin (1915-2014) with his 'suppositional logic'. Conversely, we will propose Hailperin's notion of validity (Hailperin 1996, 2011) as compatible with de Finetti approach. Finally, the differences with Reichenbach's approach will be discussed.

2. The Subordinate Event as a Logical Entity

De Finetti (1936) starts with a long critical discourse on usual three-value logic and argues in favour of a generalization of the binary formal logic of 'ordinary' events to conditional events. It presents the trivalent logic system with truth tables for the different connectors as a 'perfect analogy' to two-valued logic. De Finetti then introduces the two operations called 'thesis' and 'hypothesis' in order to return to ordinary binary logic. In de Finetti 1928a, the presentation is inverted. After a short presentation of 'subordinated event', the notions of 'thesis' and 'hypothesis' are introduced. Thus across Part 2 to Part 7, de Finetti 1928a remains in a bivalent framework. De Finetti presents the third value only in Part 7 referring to these two unary operations. Part 10 concerns the link with probability theory and Parts 11-15 constitute an arithmetic analogy of de Finetti's trivalent logic. Most of the demonstrations can be found either in the manuscript or in the various drafts that preceded it.

[4] In 1927, Bruno de Finetti, a 20 years old student in Milan, graduated in applied mathematics. Shortly after he accepted a position in Rome at the *Instituto Centrale di Statistica*, chaired at that time by the famous Italian statistician Corrado Gini Cifarelli and Regazzini 1996; de Finetti, F. and Nicotra 2008. Among these documents, there are (i) two manuscript versions (one of which is dated "Milan, March 23 1927") of de Finetti 1927, (ii) three manuscript versions (one of which is dated "Milan, May 8 1927") of de Finetti 1928d, (iii) three manuscript versions (one of which is dated, "Rome April 23 1929") which seems to be a draft of de Finetti 1932b.

[5] Capitalized by the author.

[6] These drafts will be cited here as de Finetti 1928b with their page number and identifier (#) indicated at the top of the page.

[7] In 1931, de Finetti accepted an actuary position with *Assicurazioni Generali* in Trieste (see de Finetti, F. and Nicotra 2008).

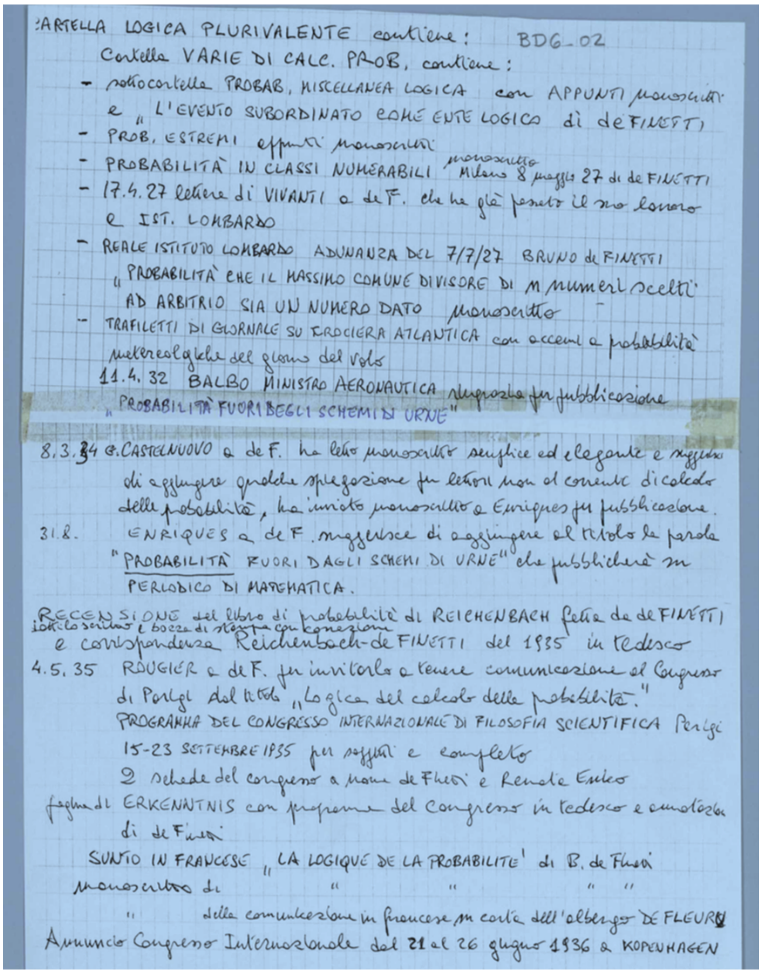

CARTELLA LOGICA PLURIVALENTE contiene: BDG-02
Cartella VARIE DI CALC. PROB. contiene:
- sottocartella PROBAB. MISCELLANEA LOGICA con APPUNTI manoscritti
e „L'EVENTO SUBORDINATO COME ENTE LOGICO di de FINETTI
- PROB. ESTREMI appunti manoscritti
- PROBABILITÀ IN CLASSI NUMERABILI manoscritto Milano 8 maggio 27 di de FINETTI
- 17.4.27 lettere di VIVANTI a de F. che ha già passato il suo lavoro e IST. LOMBARDO
- REALE ISTITUTO LOMBARDO ADUNANZA DEL 7/7/27 BRUNO de FINETTI
„PROBABILITÀ CHE IL MASSIMO COMUNE DIVISORE DI m numeri scelti AD ARBITRIO SIA UN NUMERO DATO manoscritto
- TRAFILETTI DI GIORNALE SU CROCIERA ATLANTICA con accenni a probabilità meteorologiche del giorno del volo
11.4.32 BALBO MINISTRO AERONAUTICA ringrazia per pubblicazione
„PROBABILITÀ FUORI DEGLI SCHEMI DI URNE"
8.3.34 G. CASTELNUOVO a de F. ha letto manoscritto semplice ed elegante e suggerisce di aggiungere qualche spiegazione per lettori non al corrente di calcolo delle probabilità, ha inviato manoscritto a Enriques per pubblicazione.
31.8. ENRIQUES a de F. suggerisce di aggiungere al titolo la parola "PROBABILITÀ FUORI DAGLI SCHEMI DI URNE" che pubblicherà in PERIODICO DI MATEMATICA.
RECENSIONE del libro di probabilità di REICHENBACH fatta da de FINETTI
[illegible] e bozze di stampa con correzioni
e corrispondenza Reichenbach-de FINETTI del 1935 in tedesco
4.5.35 ROUGIER a de F. per invitarlo a tenere comunicazione al Congresso di Parigi dal titolo „Logica del calcolo delle probabilità."
PROGRAMMA DEL CONGRESSO INTERNAZIONALE DI FILOSOFIA SCIENTIFICA Parigi 15-23 SETTEMBRE 1935 per sunti e completo
2 schede del congresso a nome de Finetti e Renata [illegible]
foglio di ERKENNTNIS con programma del Congresso in tedesco e annotazioni di de Finetti
SUNTO IN FRANCESE „LA LOGIQUE DE LA PROBABILITÉ' di B. de Finetti
manoscritto di " " " "
" della comunicazione in francese su carta dell'albergo DE FLEURUS
Annuncio Congresso Internazionale dal 21 al 26 giugno 1936 a KOPENHAGEN

Figure 1: Document cover of Folder 2 *"Logica Plurivalente", 1927-1935.*

2.1 The Subordinated Event as a Subordinated Bet

Part 1 corresponds to a short introduction in which de Finetti underlines the difference between the implication of propositional logic and the 'subordinate' relation used for subordinate probability. He gives the truth values for the two relations according to the 'thesis' and the 'hypothesis'. He illustrates this point by introducing the example of a bet on the outcome of a coin toss (thesis). Such a bet is subordinated to the fact that the coin has indeed been launched (hypothesis). To our knowledge, this is the first written record where de Finetti presents the conditional bet.

> The statement 'if E_2 is true then E_1 is true' of logic, in symbols: $E_2 \supset E_1$, is a true proposition if the thesis and hypothesis are true, or if the hypothesis is false, it is false only if the hypothesis is true and the thesis is false. When we speak instead of the probability of an event subordinate to another, the statement 'if E_2 is true

> then E_1 is true' has a very different value, having to be considered true if the thesis and hypothesis are true, false if the thesis is false and the hypothesis is true, and insignificant (neither true nor false) if the hypothesis is false. In fact, if one was to bet, for example, "if I throw a coin, it will show head", and then not throw the coin, one could not claim to have won the bet, although one's statement, understood as a logical deduction, is true, having a false proposition by hypothesis.
>
> Therefore we have to consider a new logical entity: the subordinate affirmation (or also subordinate event, which has by conception original and useful applications (de Finetti 1928a: 1, underlined by the author, our translation).

The subordinate event $\frac{E_1}{E_2}$ has three possible values following those of E_1 and E_2 which are 2-valued statements of the bivalent logic (a bet can only consider the realisation or non realisation of an event E). So here we have, from the outset, the idea of a three-valued logic provisionally *superimposed* on the traditional two-valued logic. The 'subordination operation' can then be extended to situations where E_1 and E_2 are 'insignificant' when they are themselves subordinated events (see section 2.6). The comparison of implication and subordinate event is given in term of 'thesis' and 'hypothesis' (subsequently noted by de Finetti 1928a T and H) which in the citation seems synonymous of consequent E_1 and antecedent E_2. The truth or falsity of the implication is given by the definition of the entailment (noted ≤) where $E_2 \leq E_1$ if $E2$ is false or if E_1 and E_2 are true (thus ≤ can often be considered as an order relation with false < true).

(1) $E_2 \supset E_1 :=: E_2 \leq E_1$

In his 1932 course at Trieste University, de Finetti also introduces the subordinate event as a conditional bet. The third value is called 'indeterminate' rather than 'insignificant'. He compares the semantic tables for implication and subordinate event in a way that we illustrate in Table 1.

> Up to now we have always talked about the probability of an E event that could only be true or false; we must now consider the more general case of an event—we shall say, of a subordinate event)—that can be either true, or false, or indeterminate.
>
> To make the concept intuitive, let us return to betting: until now we have dealt with bets made in such a way that they were certainly either won or lost; it is frequent, however, under circumstances established as indeterminate that the bet is void. For example, in a bet on the outcome of a football match, it can be agreed that the bet is void in the event of draw. In such a case, what is the betting on? The following statement: "If one of the two teams wins, the victory will go to team A".
>
> This statement differs from those considered so far in that there is a condition (that one of the two teams wins) which limits the field of possibility for which the bet is established; in it the statement 'the win is up to team A' is made subject to the condition premise, i.e. the hypothesis that 'one of the two teams wins'.
>
> Be E_1 and E_2 two events, we will generally indicate with the symbol E_1/E_2 (or also, according to the typographical convenience of the single case, $\frac{E_1}{E_2}$; read: "E_1 is subordinate to E_2") the statement (subordinate event) that "if E_2 is supposed to be true then E_1 is therefore true"; so E_1/E_2 is indeterminate when E_2 is false (i.e. in the case $-E_2$), true when E_1 and E_2 are true (case $E_1 \cap E_2$), false when E_2 is true and E_1 is false (case $E_2 \cap -E_1$).

> We must not make the confusion, despite the analogy of its verbal formulation, between the subordinate event E_1/E_2 and the event or statement that is $E_2 \supset E_1$ in formal logic ("E_2 implies E_1"), and means $(-(E_{2\cap} - E_1))$, (it is therefore false if E_1 is false and E_2 is true, while it is true in any other case, i.e. as much as when E_2 and E_1 are both true or as when E_2 is false and E_1 is either true or false). The relation between the two concepts is however close: $E_2 \supset E_1$ means "E_1/E_2 is not false" (de Finetti 1932a: 26-27, underlined by the author, our translation with the notation of de Finetti 1928a).

There is a nuance with de Finetti (1928a) in the definition of the implication. The implication is considered as an operation on the subordinate event ("E_1/E_2 is not false")[8] to pass from a ternary situation to the traditional bivalent situation. De Finetti (1928) also gives here the traditional definition 2 of implication, which is equivalent to definition 1 in the bivalent framework (see however section 3.1):

(2) $E_2 \supset E_1 :=: E_1{}^{\cup} - E_2 :=: E_{1\cap}E_2{}^{\cup} - E_2$

Both connectives are presented below in Table 1. The 'insignificant' value is the consequence of the falsehood of the hypothesis. Unlike the 'true' and 'false' values it appears as an output and not as an input of the truth table.

E_2	E_1	$E_2 \supset E_1$	$\frac{E_1}{E_2}$
true	true	true	true
true	false	false	false
false	true	true	'insignificant' or 'indeterminate'
false	false	true	'insignificant' or 'indeterminate'

Table 1: Semantic tables for implication and subordination (de Finetti 1928a,1932).

The basic idea of de Finetti, to consider that indicative conditionals can have three values of truth, was rediscovered and developed by an important number of authors with interesting variations (for a review, see the supplementary material of Baratgin, Politzer, Over et al. 2018). For example Hailperin (1996: 35-36) introduces the same Table 1 with a third value called 'don't care'.

2.2 First Definition of Subordinate Event

In Part 2, de Finetti presents the notations and symbols used. A subordinate event $E\left(\frac{E_1}{E_2}\right)$ is called 'absolute event' when E_2 is true. In this case E corresponds to the 'not-subordinated' event E_1, which can be true or false.[9]

De Finetti takes the original symbols '⊕' and '⊖' respectively for 'true' and 'false'. This choice to associate truth with a '+' and falsehood with a '−' can easily be interpreted as an implicit reference to a gain of a bet schema. If I bet on an

[8] Recall that only the situations where E_1 and E_2 are true or false are considered.

[9] This term, also used in de Finetti 1932a, will be referred as "ordinary event" in de Finetti 1936 and de Finetti 1937.

event E that comes true, I win my bet and conversely if it does not come true, E is then wrong and I lose my bet.[10]

Thus, de Finetti 1928a gives the truth and falsehood definition of a subordinate event:[11]

$$(3)\quad (^*)\quad \frac{E_1}{E_2} = \oplus \;:=:\; E_{2\cap}E_1 = \oplus \text{ and } \frac{E_1}{E_2} = \ominus \;:=:\; E_{2\cap} - E_1 = \oplus$$

It is important to stress that de Finetti at that time remains within the bi-valued framework. No reference is made to a third value or to the situation where $-E_2 = \oplus$. De Finetti, does not give any justification for the definition 3. However de Finetti (1928b: draft #'BD6-02-61', 166), partitions E_1 and E_2 as the sum of their constituents:
$E_2 = E_{1\cap}E_2{}^{\cup} - E_{1\cap}E_2 = A^{\cup}B$ and $E_1 = E_{2\cap}E_1{}^{\cup} - E_{2\cap}E_1 = A^{\cup}C$
with $A = E_{1\cap}E_2$, $B = -E_{1\cap}E_2$ and $C = -E_{2\cap}E_1$. If we suppose $E_2 = \oplus$ then $E_1 = A = -B$ which gives a correct intuition for definition 3.

2.3 Thesis, Hypothesis, Irreducible Form and Subordinate Event Equivalent

Part 3 focuses on the definition of the 'hypothesis' and 'thesis' unary operators. The 'hypothesis' (H) is the "absolute event which is necessary and sufficient to occur for a subordinate event E to be true or false" and the 'thesis' (T) is the "absolute event that is necessary and sufficient to occur for a subordinate event E to be true".[12]

$$(4)\quad H(E) \;:=:\; (E = \oplus)^{\cup}(E = \ominus) \text{ and } T(E) \;:=:\; (E = \oplus)$$

De Finetti defines the subordinated event E which follows the hypothesis and thesis:

$$(5)\quad E = \frac{E_1}{E_2} \text{ we have } H(E) = E_2 \text{ and } T(E) = E_{1\cap}E_2$$

Thus

$$(6)\quad E = \frac{T(E)}{H(E)} = \frac{E_{1\cap}E_2}{E_2}$$

Form 6 corresponds to a simplified form 'analogous to fractions reduced to the minimum terms' that de Finetti (1932a) calls 'irreducible':

[10] In latter writings, de Finetti will modify these notations by taking the traditional conventions "T" and "F" (de Finetti 1936, 1975) or Boole's convention "1" and "0" (de Finetti 1967, 1975, 1980) for true and false respectively.

[11] The * sign put by the author to identify the relation is likely to underline the importance of this relation. We added a number for the sake of identification.

[12] In de Finetti 1936 the definitions for $H(E)$ will be "E is not null" (E does not have the third truth value). Here de Finetti remains in the bivalent framework because he has not yet defined the third value.

> Let us observe that $\frac{E_1}{E_2} = \frac{A_1}{A_2}$, that is $\frac{E_1}{E_2}$ and $\frac{A_1}{A_2}$ represent the same subordinate event, if and only if $E_2 = A_2$ and $E_{2\cap}E_1 = A_{2\cap}A_1$; in fact, depending on whether one is true, false, indeterminate, the other is equally true, false, indeterminate ($-E_2 = -A_2$, $E_{2\cap}E_1 = A_{2\cap}A_1$, $E_{2\cap} - E_1 = A_{2\cap} - A_1$). Therefore it is not necessary that $E_1 = A_1$; in particular, E_1 can always be substituted with $E_{2\cap}E_1$, thus reducing the expression of the event subordinate to the form that we will say irreducible. It is a matter of eliminating also in the statement the apparent inclusion of cases that go for excluded hypothesis : if, for example. If it had been said "if one of the two teams wins, team A does not lose" (E_2 = "one of the two teams wins", E_1="team A does not lose") we would have made no more no less the same statement (possibly a bet) than before, when it was said "if one of the two teams wins, team A wins" (being $E_{2\cap}E1$ = "one of the two teams wins" and "team A does not lose"="team A wins"): the difference is only formal, because in saying "team A does not lose" the case of a draw remains included in the sentence, which, however, in the whole of the subordinate statement, remains excluded by hypothesis.
>
> In a subordinate event, or subordinate statement, $\frac{E_1}{E_2}$, one can call hypothesis the event (or statement) E_2; thesis the event (or statement $E_{2\cap}E_1$; every subordinate event can be written in the form $\left(\frac{Thesis}{Hypothesis}\right)$, and this is the form we called irreducible (de Finetti 1932a: 27-28, underlined by the author, our translation with the notations of de Finetti 1928a).

Hailperin's suppositional normal form of E (Hailperin 1996, 2011) corresponds to de Finetti's irreducible form formulated with the constituents of E from its 'condensed' semantic table by considering only the values true and false for its atoms (as in Table 1) (Hailperin 1996; Hailperin 2011). $T(E)$ corresponds to the constituents of E that are true and $H(E)$ to those that are true or false. Each event E has a 'unique suppositional normal form'. Two subordinate events are 'equivalent' if their suppositional normal forms are the same (Hailperin 1996: 250).

Which is written with the notation of de Finetti (1928) as

(7)
$$E :=: \frac{E_1}{E_2} :=: \frac{E'_1}{E'_2} :=: E' \text{ if and only if}$$
$$H(E) = E_2 :=: E'_2 = H(E') \text{ and } T(E) = E_{2\cap}E_1 :=: E'_{2\cap}E'_1 = T(E')$$

with the 'Left Logical Equivalence' principle as corollary :

(8)
$$\text{If } H(E) = E_2 :=: E'_2 = H(E') \text{ if and only if } \frac{E_1}{E_2} :=: \frac{E_1}{E'_2}$$

Part 4 is dedicated to the negation relations:

(9)
$$-E = \oplus \ :=: \ E = \ominus \text{ and } E = \ominus \ :=: \ -E = \oplus$$

De Finetti (1928a) deduces from the definition 4,[13] the hypothesis and thesis of the negation of E:[14]

[13]
$$H(-E) :=: (-E = \oplus)^{\cup}(-E = \ominus) \quad :=: (E = \ominus)^{\cup}(E = \oplus)$$
$$T(-E) :=: (-E = \oplus) :=: (E = \ominus) \quad :=: (E = \ominus)^{\cup}\left[(E = \oplus)_{\cap} - (E = \oplus)\right]$$
$$:=: \left[(E = \oplus)^{\cup}(E = \ominus)\right]_{\cap}\left[(E = \ominus)^{\cup} - (E = \oplus)\right]$$
$$T(-E) \quad :=: H(E)_{\cap} - T(E).$$

[14] The "antithesis" $-T(E)$ in de Finetti (1970) 1974: 130 is also noted $\mathcal{L}$ in de Finetti (1928b: drafts #'BD6-02-66', 171 and #'BD6-02-68', 175).

(10) $H(-E) :=: H(E)$ and $T(-E) :=: H(E)_{\cap} - T(E)$

Thus

(11) $T(E)_{\cap}T(-E) = \ominus$ and $T(E)^{\cup}T(-E) = H(E)$

2.4 Sum and Product

Part 5 focuses on the sum and product of subordinate events (formulated here for only two subordinated events E_1 and E_2).

$$(12) \begin{cases} (E_1{}^{\cup}E_2) = \oplus :=: (E_1 = \oplus)^{\cup}(E_2 = \oplus) & (E_{1\cap}E_2) = \oplus :=: (E_1 = \oplus)_{\cap}(E_2 = \oplus) \\ (E_1{}^{\cup}E_2) = \ominus :=: (E_1 = \ominus)_{\cap}(E_2 = \ominus) & (E_{1\cap}E_2) = \ominus :=: (E_1 = \ominus)^{\cup}(E_2 = \ominus) \end{cases}$$

$$(13) \begin{cases} -(E_{1\cap}E_2) = (-E_1{}^{\cup} - E_2) \\ -(E_1{}^{\cup}E_2) = (-E_{1\cap} - E_2) \end{cases}$$

The product and sum for theses and hypotheses follow:

$$(14) \begin{cases} T(E_{1\cap}E_2) = T(E_1)_{\cap}T(E_2) \\ T(E_1{}^{\cup}E_2) = T(E_1)^{\cup}T(E_2) \\ H(E_{1\cap}E_2) = [T(E_1)_{\cap}T(E_2)]^{\cup}T[-(E_{1\cap}E_2)] \\ \qquad = [T(E_1)_{\cap}T(E_2)]^{\cup}T[(-(E_1)^{\cup} - E_2)] \\ \qquad = [T(E_1)_{\cap}T(E_2)]^{\cup}T(-E_1)^{\cup}T(-E_2) \\ H(E_1{}^{\cup}E_2) = T(E_1{}^{\cup}E_2)^{\cup}[T(-E_1{}^{\cup}E_2)] \\ \qquad = T(E_1{}^{\cup}E_2)^{\cup}[T(-E_{1\cap} - E_2)] \\ \qquad = T(E_1)^{\cup}T(E_2)^{\cup}[T(-E_1)_{\cap}T(-E_2)] \end{cases}$$

De Finetti (1928a, 3) indicates 'These formulas, for $E = \frac{T(E)}{H(E)}$, give the complete expression of the sum and of the product'.[15] Thus with $E_1 = \frac{E_1'}{E_1''}$ and $E_2 = \frac{E_2'}{E_2''}$:

$$(15a) \quad \begin{aligned} E_{1\cap}E_2 &= \frac{T(E_{1\cap}E_2)}{H(E_{1\cap}E_2)} \\ &= \frac{T(E_1)_{\cap}T(E_2)}{[T(E_1)_{\cap}T(E_2)]^{\cup}T(-E_1)^{\cup}T(-E_2)} \\ &= \frac{E_1'{}_{\cap}E_1''{}_{\cap}E_2'{}_{\cap}E_2''}{[E_1'{}_{\cap}E_1''{}_{\cap}E_2'{}_{\cap}E_2'']^{\cup}[-E_1'{}_{\cap}E_1''{}^{\cup} - E_{2\cap}'E_2'']} \end{aligned}$$

[15] The formulation 15b comes from de Finetti 1928b: draft #'BD6-02-67', 173. These formulas were rediscovered much later independently by Goodman, Nguyen and Walker (1991) and by Hailperin (1996).

$$
\begin{aligned}
&:=: \frac{E'_{1\cap}E''_{1\cap}E'_{2\cap}E''_2}{E''_{1\cap}E''^{\cup}_2 - E'_{1\cap}E''^{\cup}_1 - E_{2\cap}{}'E''_2} \\
&:=: \frac{E'_{1\cap}E'_2}{E''_{1\cap}E''^{\cup}_2 - E'_{1\cap}E''^{\cup}_1 - E_{2\cap}{}'E''_2}
\end{aligned}
$$

$$
\begin{aligned}
E_1{}^{\cup}E_2 &= \frac{T(E_1{}^{\cup}E_2)}{H(E_1{}^{\cup}E_2)} \\
&= \frac{T(E_1)^{\cup}T(E_2)}{T(E_1)^{\cup}T(E_2)^{\cup}\,[T(-E_1)_{\cap}T(-E_2)]} \\
\text{(15b)}\qquad &= \frac{E'_{1\cap}E''^{\cup}_1E'_{2\cap}E''_2}{[E'_{1\cap}E''^{\cup}_1E'_{2\cap}E''_2]^{\cup}\,[-E'_{1\cap}E''_{1\cap} - E'_{2\cap}E''_2]} \\
&:=: \frac{E'_{1\cap}E''^{\cup}_1E'_{2\cap}E''_2}{E'_{1\cap}E''^{\cup}_1E'_{2\cap}E''^{\cup}_2E''_{1\cap}E''_2} \\
&:=: \frac{E'^{\cup}_1E'_2}{E'_{1\cap}E''^{\cup}_1E'_{2\cap}E''^{\cup}_2E''_{1\cap}E''_2}
\end{aligned}
$$

De Finetti poses the 'well-known logical identities'[16]

$$
\text{(16)}\quad
\begin{cases}
(E_{1\cap}E_2)^{\cup}E_3 = (E_1{}^{\cup}E_3)_{\cap}(E_2{}^{\cup}E_3) \\
(E_1{}^{\cup}E_2)_{\cap}E_3 = (E_{1\cap}E_3)^{\cup}(E_{2\cap}E_3)
\end{cases}
$$

In Part 6 de Finetti affirms that the hypotheses of the sum and the product of subordinate events are always contained (noted $:\supset:$) between the sum of the hypotheses and their product.[17]

$$
\text{(17)}\quad
\begin{cases}
H(E_1)_{\cap}H(E_2) \;:\supset:\; H(E_{1\cap}E_2) \;:\supset:\; H(E_1)^{\cup}H(E_2) \\
H(E_1)_{\cap}H(E_2) \;:\supset:\; H(E_1{}^{\cup}E_2) \;:\supset:\; H(E_1)^{\cup}H(E_2)
\end{cases}
$$

with the corollary:

$$\text{if } H(E_1) = H(E_2) = E \text{ then } H(E_1)_{\cap}H(E_2) = H(E_1)^{\cup}H(E_2) = E.$$

[16] The demonstration can be read in de Finetti 1928b: draft #'BD6-02-65', 170:

$$
\begin{aligned}
T\left[(E_{1\cap}E_2)^{\cup}E_3\right] &= T(E_{1\cap}E_2)^{\cup}T(E_3) = [T(E_1)_{\cap}T(E_2)]^{\cup}T(E_3) \\
&= \left[T(E_1)^{\cup}T(E_3)\right]_{\cap}\left[T(E_2)^{\cup}T(E_3)\right] \\
T\left[(E_{1\cap}E_2)^{\cup}E_3\right] &= T(E_{1\cap}E_2)^{\cup}T(E_3) = T\left[(E_1{}^{\cup}E_3)_{\cap}(E_2{}^{\cup}E_3)\right] \\
T\{-\left[(E_{1\cap}E_2)^{\cup}E_3\right]\} &= T\left[-(E_{1\cap}E_2)^{\cup}T(-E_3)\right] = [T(-E_1)_{\cap}T(-E_2)]_{\cap}T(-E_3) \\
&= \left[T(-E_1)^{\cup}T(-E_2)\right]_{\cap}T(-E_3) \\
T\{-\left[(E_{1\cap}E_2)^{\cup}E_3\right]\} &= T(-E_{1\cap} - E_3)^{\cup}T(-E_{2\cap} - E_3).
\end{aligned}
$$

[17] The demonstration is in de Finetti 1928b: draft #'BD6-02-66', 171: As $T(E_1)_{\cap}T(E_2) :\supset: T(E_1)^{\cup}T(E_2)$

$$
\begin{aligned}
H(E_1{}^{\cup}E_2) &= T(E_1)^{\cup}T(E_2)^{\cup}\,[T(-E_1)_{\cap}T(-E_2)] :\supset: T(E_1)^{\cup}T(E_2)^{\cup}T(-E_1)^{\cup}T(-E_2) \\
&= H(E_1)^{\cup}H(E_2) \\
H(E_{1\cap}E_2) &= H(-E_{1\cap} - E_2) :\supset: H(-E_1)^{\cup}H(-E_2) = H(E_1)^{\cup}H(E_2) = H(E_1{}^{\cup}E_2).
\end{aligned}
$$

Thus de Finetti deduces 'the simple formulas':

$$(18)\quad \begin{cases} \dfrac{E_1}{E_3} {}^{\cup} \dfrac{E_2}{E_3} = \dfrac{E_1 {}^{\cup} E_2}{E_3} \\ \dfrac{E_1}{E_3} {}_{\cap} \dfrac{E_2}{E_3} = \dfrac{E_1 {}_{\cap} E_2}{E_3} \end{cases}$$

Thus

$$(19)\quad \frac{E_1}{E_2} {}_{\cap} \frac{E_2}{E_3} = \frac{E_1}{E_3}$$

In Part 7 de Finetti shows that an absolute event is a particular subordinate event characterized by any one of the following conditions:

$$(20)\quad \begin{cases} T(E) = E \\ T(-E) = -E \\ H(E) = \oplus \end{cases}$$

that is $E = \frac{E}{\oplus}$. In particular $\oplus = \frac{\oplus}{\oplus}$, and $\ominus = \frac{\ominus}{\oplus}$. As mentioned in 1932:

> We again observe that an absolute event (that we can call, to distinguish it, an event of the type previously considered) can be considered as a particular case of subordinate event (precisely that case in which the 'hypothesis' is a certain event) (de Finetti 1932a: 28, underlined by the author, our translation).

2.5 The Third Truth Value: Insignificant

The third value 'insignificant' is introduced in Part 8. A subordinate event $E = \frac{E_1}{E_2}$ is insignificant (noted $\odot$) when the hypothesis is false ($H(E) = E_2 = \ominus$). In this case the thesis is also false ($T(E) = E_1 {}_{\cap} \ominus = \ominus$). Unlike the 'true' and 'false' values it is a 'transitory' value.

$$(21)\quad \begin{cases} \odot = \dfrac{\ominus}{\ominus} \\ T(\odot) = H(\odot) = \ominus \\ -\odot = \odot \\ H(-\odot) = H(\odot) = \ominus \\ T(-\odot) = H(\odot)_{\cap} - T(\odot) = \ominus_{\cap} - \ominus = \ominus \end{cases}$$

Conversely if $-X = X$ then $X = \odot$ ($T(-X) = T(X)$ thus $T(X) = T(X) {}_{\cap} T(-X) = \ominus$, $T(-X) = \ominus$, $H(X) = T(X)_{\cap} T(-X) = \ominus$ and $X = \frac{\ominus}{\ominus} = \ominus$). To finish de Finetti gives[18] the relations 22:

[18] The demonstration is in de Finetti 1928c: $T(E^{\cup}\odot) = T(E)$, and $T(-(E^{\cup}\odot)) = T(-E_{\cap}\odot) = \ominus$, thus $H(E^{\cup}\odot) = T(E)$ and $E^{\cup}\odot = \frac{T(E)}{T(E)} = \frac{\oplus}{T(E)}$. Likewise $T(E_{\cap}\odot) = \ominus$ and $T(-(E_{\cap}\odot)) = T(-E^{\cup}\odot) = T(-E)$, thus $H(E_{\cap}\odot) = T(-E)$ and $E_{\cap}\odot = \frac{T(-E)}{T(-E)} = \frac{\ominus}{T(-E)}$.

$$(22)\quad \begin{cases} E^{\cup}\odot = \dfrac{\oplus}{T(E)} \\ E_{\cap}\odot = \dfrac{\ominus}{T(-E)} \end{cases}$$

It is easy[19] to write $E = \frac{E_1}{E_2}$ in a similar manner as the definition of the implication given in 2 (see section 2.1) :

$$(23)\quad E = \frac{T(E)}{H(E)} :=: (T(E)_{\cap}H(E))^{\cup}(\odot_{\cap} - H(E)) :=: (E_{1\cap}E_2)^{\cup}(-E_{2\cap}\odot)$$

This definition corresponds to that of the suppositional connective of Hailperin (see 1996: 36).

These relations allow to clarify the values for the different unary operators (see Table 2).

E	$T(E)$	$H(E)$	$T(-E)$	$H(-E)$	$-T(E)$	$-H(E)$
$\oplus$	$\oplus$	$\oplus$	$\ominus$	$\oplus$	$\ominus$	$\ominus$
$\odot$	$\ominus$	$\ominus$	$\ominus$	$\ominus$	$\oplus$	$\oplus$
$\ominus$	$\ominus$	$\oplus$	$\oplus$	$\oplus$	$\oplus$	$\ominus$

Table 2: Semantic tables for unary operations.

The thesis and hypothesis have been independently rediscovered by several authors. As noted by Mura 2009, the thesis T corresponds to the 'external' connector of Bochvar (1937) 1981. Recently Blamey 2001, Cantwell 2006, Lassiter 2020 introduce both unary operators T and H with different notations. Montagna (2012) sets out three operations which correspond to thesis T ('E is true'), antithesis $-T$ ('E is false') and anti-hypothesis$-H$ ('E is insignificant').

2.6 The Extension to Nested Subordinate Events

Having defined the truth value 'insignificant', de Finetti extends in Part 9 the $\frac{E_1}{E_2}$ subordination operation to the case where E_1 and E_2 are subordinate events with three possible values. He starts by defining T and H.[20]

$$(24)\quad \begin{cases} T\left(\dfrac{E_1}{E_2}\right) = T(E_{1\cap}E_2) = T(E_1)_{\cap}T(E_2) \\ \\ H\left(\dfrac{E_1}{E_2}\right) = T(E_{2\cap}E_1)^{\cup}(T(E_{2\cap} - E_1)) \\ \qquad = [T(E_2)_{\cap}T(E_1)]^{\cup}[T(E_2)_{\cap}T(-E_1)] \\ \qquad = T(E_2)_{\cap}[T(E_1)^{\cup}T(-E_1)] = T(E_2)_{\cap}H(E_1) \end{cases}$$

[19] $E = E_{1\cap}E_2 = T(E)$ when $H(E) = \oplus$ and $E = \odot$ when $(-E_2) = -H(E) = \oplus$, thus $E = (T(E)_{\cap}H(E))^{\cup}(\odot_{\cap} - H(E))$.

[20] The relation $T\left(\frac{E_1}{-E_2}\right) = \bot\left(\frac{E_1}{E_2}\right) = T(E_1)_{\cap}T(-E_2)$ is in de Finetti 1928b: draft #'BD6-02-68', 175. These relations will be rediscovered by Montagna 2012.

Thus one can write $\frac{E_1}{E_2}$ in its 'reduced form':

$$(25)\quad \frac{E_1}{E_2} = \frac{T(E_1)\cap T(E_2)}{T(E_2)\cap H(E_1)}$$

And also in the 'subordinate form' by taking $E_1 = \frac{E_1'}{E_1''}$ and $E_2 = \frac{E_2'}{E_2''}$:

$$(26)\quad \frac{E_1}{E_2} = \frac{\frac{E_1'}{E_1''}}{\frac{E_2'}{E_2''}} :=: \frac{E_1'\cap E_1''\cap E_2'\cap E_2''}{E_1''\cap E_2'\cap E_2''} :=: \frac{E_1'}{E_1''\cap E_2'\cap E_2''}$$

The relation 26, called the 'Import-Export law' in the literature, will appear in appendix of de Finetti (1970) 1975: 328. It entails the following corollaries (Hailperin 1996: 253):

$$(27)\quad \begin{cases} \frac{\frac{E_1'}{E_1''}}{E_2} :=: \frac{E_1'}{E_1''\cap E_2} \\ \frac{E_1'}{\frac{E_1''}{E_2}} :=: \frac{E_1'}{E_1''\cap E_2} \\ \frac{\frac{E_1'}{E_1''}}{\frac{E_2'}{E_2''}} :=: \frac{E_1'\cap E_1''}{E_2'\cap E_2''} \end{cases}$$

Thus for example for all event E

$$(28)\quad \frac{E}{\odot} = \odot$$

The important consequence of relations 15a, 15b and 26 is that all events E comprising some subordinate events, with the fraction symbol, may be written in a single subordinate form.

2.7 The Level of Probability

De Finetti (1928a: 5) explains that "The logical operations introduced allow the symbolic writing of theorems on the subordinate probabilities", such as for example:

$$(29)\quad P(E) = \frac{P[T(E)]}{P[H(E)]} \quad (\text{with } P[H(E)] \neq 0)$$

De Finetti also gives, with E_1 and E_2 absolute event, the definition of conditional probability and axioms recently rediscovered by some authors (Cantwell 2006, Mura 2009, Rothschild 2014, Lassiter 2020).

$$(30)\quad \begin{cases} P(E_1{\cap}E_2) & = P(E_2) \times P\left(\frac{E_1}{E_2}\right) \\ P\left(\frac{E_1{\cup}E_2}{E_3}\right) & = P\left(\frac{E_1}{E_3}\right) + P\left(\frac{E_2}{E_3}\right) \quad (\text{with } E_1{\cap}E_2{\cap}E_3 = \ominus) \\ P\left(\frac{E}{E}\right) & = 1 \\ P\left(\frac{-E}{E}\right) & = 0 \end{cases}$$

The proof of the theorem of conditional probability can be found in de Finetti 1928b: draft #‘BD06-02-69’, 164 (see the Table 3) and is also discussed in de Finetti 1932a: 29-30.

$\frac{E_1}{E_2}$		E_2		
		$\oplus$	$\odot$	$\ominus$
	$\oplus$	$\oplus 1$	$\odot 2$	$\odot 3$
E_1	$\odot$	$\odot 4$	$\odot 5$	$\odot 6$
	$\ominus$	$\ominus 7$	$\odot 8$	$\odot 9$

$+S_1$	$(1,2,3)$	
$+S_2$	$(1,4,7)$	E_1E_2 1
$+S$	(1)	$-E_1E_2$ 7
$-pS$	$(1,7)$	$-E_1-E_2$ 9
$-pS_1$	$(1,2,3,7,8,9)$	
$-pS_2$	$(1,4,7,3,6,9)$	$H(E_1)H(E_2) = (1,3,7,9)$

		S	S_1	S_2
*	$G_1 =$	$1-p$	$1-P_1$	$1-P_2$
	$G_2 =$	0	$1-P_1$	0
*	$G_3 =$	0	$1-P_1$	$-p_2$
	$G_4 =$	0	0	$1-p_2$
	$G_5 =$	0	0	0
	$G_6 =$	0	0	$-P_2$
*	$G_7 =$	$-p$	$-p_1$	$-P_2$
	$G8 =$	0	$-p_1$	0
*	$G9 =$	0	$-p_1$	$-p_2$

$$\begin{vmatrix} 1-p & 1-P_1 & 1-P_2 \\ -p & -p_1 & -P_2 \\ 0 & -p_1 & -p_2 \end{vmatrix} = \begin{vmatrix} 0 & 1 & 1 \\ -p & -p_1 & 1-p_2 \\ 0 & -p_1 & -p_2 \end{vmatrix}$$

$$= \begin{vmatrix} 0 & 1 & 1 \\ -p & 0 & -p_1 \\ 0 & -p_1 & -p_2 \end{vmatrix} = p_1 - pp_2 \qquad p = \frac{p_1}{p_2}$$

With $p = P\left(\frac{E_1}{E_2}\right)$, $p_1 = P(E_1)$ and $p_2 = P(E_2)$.

Table 3: De Finetti’s proof of theorem 24 (colored in red by the author).

It corresponds to the three first stages of the demonstration in de Finetti (1937: 14) (see section 1.). Here, the matching between stages (i) and (ii) are more detailed. The truth table of subordinate event $E{=}\frac{E_1}{E_2}$ clarifies the nine possible gains $G_1,...,G_9$ corresponding to nine possible values of E. The gains G_1, G_3, G_7, G_9, marked with an *, correspond to a situation where the conjunction of hypothesis of E_1 and E_2 is true—$H(E_1){\cap}H(E_2) = (G_1, G_3, G_7, G_9)$. The bet on $\frac{E_1}{E_2}$ is envisaged only in the case where its constituents E_1 and E_2 are not insignificant (see section 2.8). Now as it is supposed that $E_1 \supset E_2$, the case G_3 should not be considered. The three gains G_1, G_7, G_9, marked with a red *, correspond to the three possible cases E_1, $E_{2\cap} - E_1$ and $-E_2$. The coherence constraint implies that the determinant of the linear equation system CP must be null—stage (iii). Since $E_1 \supset E_2$, then $P\left(\frac{E_1}{E_2}\right) = \frac{P(E_1)}{P(E_2)}$. The transition to the general case $E_{1\cap}E_2$ in place of E_1 (when $E_1 \supset E_2$) (stage iv) is not mentioned. In de Finetti 1932a: 30 this transition is justified: “the difference is only external, and depends on the fact that when a subordinate event is expressed in its irreducible form (as in the said example), E_1 and $E_{1\cap}E_2$ are the same thing”.

De Finetti defines the notion of independence:

$$(31)\quad \begin{cases} E_1 \text{ independent of } E_2 :=: P\left(\dfrac{E_1}{E_2}\right) = P(E_1) \\ \text{So} \\ P(E_2) = P\left(\dfrac{E_2}{E_1}\right), \\ P(E_{1\cap}E_2) = P(E_1) \times P(E_2) \end{cases}$$

2.8 An Arithmetical Analogy

To finish, in Parts 11 to 14, de Finetti (1928a) introduces a 'remarkable arithmetic analogy'. The event (absolute or subordinate) E, is considered as a random variable x that takes the value $+1$, 0, -1 depending on whether E is true (⊕), insignificant (⊙) or false (⊖). The relation with gains of a conditional bet is obvious even if it is not made explicitly. Each value corresponds to a payoff according to the three possible consequences of a bet on E. If E becomes true, one wins 1€ (2€ − 1€ = 2€$(1-\frac{1}{2}) = S(1-p)$), if E is false, one loses 1€ ($-\frac{1}{2}$2€ $= -pS$) and if E is insignificant one gets back the stake −1€ + 1€ = 0€ $= pS - pS$. Such bet corresponds to the degree of indifference to bet on E or $-E$ equal to $\frac{1}{2}$. This corresponds to Ramsey (1926) 1999's definition of 'ethically neutral proposition'. Thus, the fact that the bettor agrees to bet on E (while he is indifferent between E and $-E$, also amounts to agreeing to bet on the C_i constituents of E that give E true (while the bettor is also indifferent between C_i and $-C_i$). In other words, de Finetti's logic corresponds to a first epistemic level (Baratgin and Politzer 2016), where an individual evaluates the truth or falsity of E (without preference) in the same way a bettor specifies the terms of a bet on the E event in considering only the bi-valued of its constituents (the different possible $(-1, +1)$-model in the semantic table of E reduced to this bi-valued model). A bet is possible if at least a $(-1, +1)$-model gives the value 1. This 'indifferent' step is necessary as a first step in order to elaborate a probability judgment on E (the outlaid pay) which corresponds to the second epistemic level (de Finetti 1980, Baratgin and Politzer 2016).

$$(32)\quad \begin{cases} E = \oplus \ :=: \ x = +1 \\ E = \odot \ :=: \ x = 0 \\ E = \ominus \ :=: \ x = -1 \end{cases}$$

Considering the random variables x_1, x_2 for E_1, E_2, the random variables for $E_1{}^\cup E_2$ and $E_{1\cap}E_2$ correspond to respectively the $\max(x_1, x_2)$ and the $\min(x_1, x_2)$. Hailperin (1996) later formulates the same relations. Hence, it is possible to rearrange in an ascending order the three truth values. Figure 2 represents the three truth values as a function of both the level of knowledge (K) (with ⊙ for the ignorance) and the level of gain (G). As noted by Mura (2009), it is the truth-valued gap interpretation of partial logic (e.g. Blamey 2001).

De Finetti points out that the negation corresponds to the multiplication by -1. $-E$ corresponds to the random variable x':

$$(33)\quad x' = -x$$

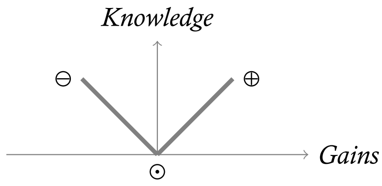


Figure 2: Representation of three truth values according to levels of knowledge and gains.

Let E_1, E_2 be absolute or subordinated events corresponding to random variables x_1, x_2, and $\frac{E_1}{E_2}$ with random variable x, we have:

(34) $x = \dfrac{x_1 x_2 (1 + x_2)}{2}$

Thus if E_2 is false, $x_2 = -1$ and $x = 0$, if E_2 is insignificant, $x_2 = 0$ and $x = 0$. In both cases E is insignificant. Now if E_2 is true, $x_2 = 1$ and $x = x_1$.

Finally the thesis $T(x)$ and hypothesis $H(x)$ are written as:

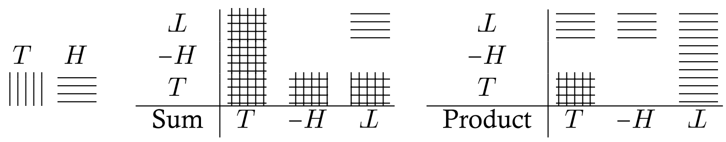
(35) $T(x) = x^2 + x - 1$ and $H(x) = 2x^2 - 1$

3. De Finetti's 2-to-3 Valued Logic

3.1 The First Truth Tables of de Finetti's Trivalent System

Two drafts in de Finetti 1928b explicitly show the tables of truth for negation, product, sum and subordination in the way de Finetti 1936 will present them while de Finetti 1928a, describes them, albeit not explicitly.

The truth tables for the 'sum' and 'product' of tri-events are illustrated in de Finetti 1928b (draft #'BD6-02-66': 171) with as input the thesis T, the anti-hypothesis $-H$ and the antithesis ($Ʇ$) (Table 4).



With T thesis, $-H$ anti hypothesis and $Ʇ$ anti thesis.

Table 4: De Finetti's tables for sum and product, following thesis, anti-hypothesis and anti-thesis (from de Finetti 1928b, draft #'BD6-02-66': 171).

By replacing the squared cells ($T = H = ⊕$) by ⊕, the empty cells ($H = T = ⊖$) by ⊙ and the cells with horizontal line ($T = ⊖$ and H=⊕) by ⊕ we obtain the tables for the sum and product of subordinate event.

The truth tables for product, sum, subordination and implication connectives (see Table 5)[21] are illustrated in de Finetti (1928b: draft #'BD6-02-58', 163).

[21] For the sake of consistency with section 2.1, E_1 and E_2 are swapped for E_2 and E_1.

$E_1 = E_{1\cap}E_2{}^{\cup}E_{1\cap} - E_2$

$E_{1\cap}E_2$		E_1		
		$\oplus$	$\odot$	$\ominus$
	$\oplus$	$\oplus$	$\odot$	$\ominus$
E_2	$\odot$	$\odot$	$\odot$	$\ominus$
	$\ominus$	$\ominus$	$\ominus$	$\ominus$

$E_1{}^{\cup}E_2$		E_1		
		$\oplus$	$\odot$	$\ominus$
	$\oplus$	$\oplus$	$\oplus$	$\oplus$
E_2	$\odot$	$\oplus$	$\odot$	$\odot$
	$\ominus$	$\oplus$	$\odot$	$\ominus$

$\frac{E_1}{E_2}$		E_1		
$(E_{1\cap}E_2)^{\cup} - E_{2\cap}\odot$		$\oplus$	$\odot$	$\ominus$
	$\oplus$	$\oplus$	$\odot$	$\ominus$
E_2	$\odot$	$\odot$	$\odot$	$\odot$
	$\ominus$	$\odot$	$\odot$	$\odot$

$E_2 \supset_1 E_1$		E_2		
$E_2 \leq_{\odot} E_1$		$\oplus$	$\odot$	$\ominus$
	$\oplus$	$\oplus$	$\oplus$	$\oplus$
E_1	$\odot$	$\ominus$	$\oplus$	$\oplus$
	$\ominus$	$\ominus$	$\ominus$	$\oplus$

(de Finetti 1936: 35)

$E_2 \supset_2 E_1$		E_2		
$(E_{1\cap}E_2)^{\cup} - E_2$		$\oplus$	$\odot$	$\ominus$
	$\oplus$	$\oplus$	$\oplus$	$\oplus$
E_1	$\odot$	$\odot$	$\odot$	$\oplus$
	$\ominus$	$\ominus$	$\odot$	$\oplus$

Table 5: De Finetti's truth tables for product, sum, subordinate, and implication from de Finetti 1928b, draft #'BD6-02-58': 163. In gray, the subordinate and implications definitions.

The truth table for implication is not the one that will be given in de Finetti (1936) (noted respectively '$\supset_1$' and '$\supset_2$'). De Finetti defines it in the same page:

(36) $E_2 \supset_1 E_1 :=: E_{2\cap}E_1 = E_1$

He demonstrates[22] that:

(37) $E_2 \supset_1 E_1 :=: T(E_2) \supset_1 T(E_1)$ and $T(-E_1) \supset_1 T(-E_2)$.

Now setting E_1 and E_2 as $\frac{E_1'}{E_1''}$ and $\frac{E_2'}{E_2''}$ (thus $T(E_1) = E'_{1\cap}E''_1$ and $T(E_2) = E'_{2\cap}E''_2$), relation 37 yields the relation of implication from unconditional events to conditional events discovered by Goodman and Nguyen (1988):

(38) $\frac{E_2'}{E_2''} \supset \frac{E_1'}{E_1''} :=: E'_{2\cap}E''_2 \supset E'_{1\cap}E''_1$ and $-E'_{1\cap}E''_1 \supset -E'_{2\cap}E''_2$

The implication $\supset_1$ is not equivalent to $-(E_{2\cap}-E_1) :=: E_1{}^{\cup}-E_2 :=: (E_{1\cap}E_2)^{\cup}-E_2$ contrarily to $\supset_2$.[23] Each of these two truth tables corresponds respectively to the generalization to trivalent cases to both definitions 1 and 2 given by de Finetti (1928a) and de Finetti (1932a) (see section 2.1). The implication $\supset_1$ can be defined following the entailment (noted $\leq_{\odot}$) generalizing the bivalued entailment $\leq$ assuming the natural order that $\ominus$ is less true than $\odot$ and $\odot$ is less true than $\oplus$. So $E_2 \leq_{\odot} E_1$ if the value of E_1 is at least as strong as the value of E_2. This order and entailment is supported by some authors (e.g. Milne 1997; Mura 2009; Hailperin 2011; Vidal 2014). The implication $\supset_2$ respects the traditional equivalence to $-(E_{2\cap} - E_1)$.[24] Rescher 1969: 46-52, independently, in order to generalize the bivalent logic system will propose, for the same reason, both successive

[22] if $E_2 \supset_1 E_1$, $E_{2\cap}E_1 = E_1$, thus $T(E_{2\cap}E_1) = T(E_1)$ and also $T(E_{2\cap}E_1) = T(E_2)_{\cap}T(E_1) = T(E_1)$ thus $T(E_2) \supset_1 T(E_1)$. $T(-(E_{2\cap}E_1)) = T(-E_1)$ and $H(E_{2\cap}E_1) = H(E_1)$. Also $T(-(E_{2\cap}E_1)) = T(-E_{2\cap} - E_1) = T(-E_2)_{\cap}T(-E_1) = T(-E_1)$, hence $T(-E_1 \supset_1 -E_2)$.

[23] For the three following cases: $\odot \supset_1 \ominus :=: \ominus$, $\odot \supset_1 \odot :=: \oplus$ and $\ominus \supset_1 \odot :=: \ominus$, while with $\supset_2$, we obtain $\odot$. However $\supset_1$ respects the equivalence $(E_2 \supset_1 E_1)_{\cap}E_2 :=: E_{2\cap}E_1$ while as noted by Égré, Rossi and Sprenger 2020b $(\odot \supset_2 \ominus)_{\cap}E_2 = \odot$ and $\odot_{\cap}\ominus = \ominus$.

[24] It is certainly for this reason that de Finetti modifies the implication table as from 1932.

systems S_3 and K_3 (Kleene' system) which correspond (without the subordinate connective) to both of de Finetti's systems.

3.2 Validity for de Finetti's 2-to-3 Valued Logic

In bi-valued logic, an event E is valid (noted $\therefore E$) if its value is ⊕ under all possible assignments of truth values to its atomic components. An argument E_2 then E_1 is valid (noted $E_2 \therefore E_1$) if it preserves the truth of its premises. That is, if there is no model that renders a premise true and the conclusion false. In the bet analogy, a valid event can be interpreted as a sure bet and the valid inference as a bet preservation it is not possible to bet that E_2 is true without betting that E_1 is true. As it has been pointed out in numerous occasions (see sections 2.1, 2.5, 2.6, 2.7 and 2.8) de Finetti' system corresponds to a logic superimposed on the bivalent logic (i.e a '2-to 3-valued logic'). In generalising the bet analogy, confronted to a subordinate bet, the bettor assigns only the values ⊖ and ⊕ to all the possible atomic components of E (⊖, ⊕)-model in the restricted truth table (the situation where $H(E_1)_\cap H(E_2)$ is true (e.g. see the Tables 1 and 3 restraint to (G_1, G_3, G_7, G_9)). Thus the bettor analyses a 'condensed true table'.[25] Thus

- An event E is ⊙-valid (noted $\therefore_\odot E$) if there is no (⊖, ⊕)-model for which the value in E is ⊖ and if there is at least one model for which its value is ⊕. Concretely if the value column of its semantic restraint table has at least one occurrence of ⊕ and no occurrences of ⊖.
- An argument $E_2 \therefore E_1$ is ⊙-valid (noted $E_2 \therefore_\odot E_1$) if it 'preserves the ⊙-validity'.

This definition of ⊙-validity has been proposed by Hailperin (1996: 35-36 and 246-253), who also took the ⊙-entailment definition Hailperin (2011: 33-34).

The Tables 6 and 7 expose some principles between subordination, implication,[26] ⊙-validity and ⊙-entailment and traditional events and arguments.

The subordination respects the ideal trilemma (Identity, Modus-Ponens and non symmetry) required by Égré, Rossi and Sprenger (2020a). However it does not collapse the implication $\left(E_2 \supset E_1 \not\therefore_\odot \frac{E_1}{E_2}\right)$ since Supraclassicality fails although all other properties (Import-Export (26), Left Logical Equivalence (10), stronger-than-implication (R2)) are satisfied.

[25] It is therefore very important to consider the de Finetti's system as a 2-to-3 valued logic and not as the traditional interpretation of three valued logic (e.g. Mura 2009, Vidal 2014, Égré, Rossi and Sprenger 2020a). At the probabilistic level, the failure to take into account the restricted form leads to incoherence (see Cantwell 2006).

[26] With "⊃" for bivalent implication.

Hailperin (1996: 248)	R_1 If $\frac{E_1}{E_2}$ is $\odot$-valid then $E_2 \supset E_1$ is valid	
[1]	R_2 Stronger-than-implication	$\frac{E_1}{E_2} \therefore_{\odot} E_2 \supset E_1$
Hailperin (1996: 248)	R_3 Absolute event validity	An absolute event E is $\odot$-valid, if and only if it is valid
[2]	R_5 Entailment versus $\odot$-validity	If $E_2 \leq E_1$ and if there is at least a model that gives E_2 true, then $E_2 \therefore_{\odot} E_1$ and $\therefore_{\odot} \frac{E_1}{E_2}$
[3]	R_6 no $\odot$-validity versus no Entailment	If $E_2 \not\therefore_{\odot} E_1$, then $E_2 \not\leq_{\odot} E_1$
[4]	R_7 Conditional Elimination fails	~~If $\therefore E_2 \supset E_1$ then $E_2 \therefore_{\odot} E_1$~~
Similar to [4]	R_8 Supraclassicality fails	~~If $E_2 \therefore E_1$ then $\therefore_{\odot} \frac{E_1}{E_2}$~~

[1] $E_2 \supset E_1$ is false when $E_2 = \oplus$ and $E_1 = \ominus$. In this case $\frac{E_1}{E_2}$ is also false.
[2] In model where $E_1 = \ominus$ then E_2 also, and in some model where $E_1 = \odot$ then $E_2 = \ominus$ or $E_2 = \odot$. In add there are at least a model that gives E_1 true.
[3] $E_2 \not\therefore_{\odot} E_1$ if (i) $E_1 = \ominus$ and $E_2 = \odot$ or (ii) $E_1 = \odot$ and $E_2 = \ominus$. in these both cases $E_2 \not\leq_{\odot} E_1$.
[4] Take $E_2' = \ominus$ and $E_1' = \oplus$.

Table 6: Relations between implication and subordination connectives.

Proofs	**Events**	**Epistemic belief level**		**Epistemic degree of belief level**[1]
Hailperin (1996: 249)[3]	Identity	$\therefore_{\odot} \frac{A_1}{A_1}$		$P\left(\frac{A_1}{A_1}\right) \in [0,1]$
Hailperin (1996: 249)		$\therefore_{\odot} \frac{A_1 \cup A_2}{A_1}$		$P\left(\frac{A_1 \cup A_2}{A_1}\right) \in [0,1]$
Hailperin (1996: 249)		$\therefore_{\odot} \frac{A_1}{A_1 \cap A_2}$		$P\left(\frac{A_1}{A_1 \cap A_2}\right) \in [0,1]$
Hailperin (1996: 249)		$\not\therefore_{\odot} \frac{E}{A_1 \cap -A_1}$		$P\left(\frac{C}{A \cap -A}\right) \in [0,1]$
	Arguments			
R_3	And-introduction	$E_2, E_1 \therefore_{\odot} E_2 \cap E_1$	$\leq_{\odot}$	$P(E_2 \cap E_1) \in [\max\{0, P(E_2) + P(E_1) - 1\}, \min\{P(E_2), P(E_1)\}]$
R_3	And-elimination	$E_2 \cap E_1 \therefore_{\odot} E_2$	$\leq_{\odot}$	$P(E_2) \in [P(E_2 \cap E_1), 1]$
R_3	Or-introduction	$E_2, E_1 \therefore_{\odot} E_2 \cup E_1$	$\leq_{\odot}$	$P(E_2 \cup E_1) \in [\max\{P(E_2), P(E_1)\}, \min\{P(E_2) + P(E_1), 1\}]$
R_3		$E_2 \therefore_{\odot} E_2 \cup E_1$	$\leq_{\odot}$	$P(E_2 \cup E_1) \in [P(E_2), 1]$
R_3	If-introduction	$E_2, E_1 \therefore_{\odot} \frac{E_1}{E_2}$	$\leq_{\odot}$	$P\left(\frac{E_1}{E_2}\right) \in \left[\max\left\{0, \frac{(P(E_1)+P(E_2)-1)}{P(E_2)}\right\}, \min\left\{\frac{P(E_1)}{P(E_2)}, 1\right\}\right]$
[3]	Consequent to 'if'	$E_1 \therefore_{\odot} \frac{E_1}{E_2}$	$\not\leq_{\odot}$	$P\left(\frac{E_1}{E_2}\right) \in [0,1]$
Verification of $\leq_{\odot}$ and R_5	'And' to 'if'	$E_2 \cap E_1 \therefore_{\odot} \frac{E_1}{E_2}$	$\leq_{\odot}$	$P\left(\frac{E_1}{E_2}\right) \in [P(E_2 \cap E_1), 1]$
Similar to [3]	'Or' to 'if not'	$E_2 \cup E_1 \therefore \frac{E_1}{-E_2}$	$\not\leq_{\odot}$	$P\left(\frac{E_1}{-E_2}\right) \in [0, P(E_2 \cup E_1)]$
Hailperin (1996: 248)	Modus Ponens	$\frac{E_1}{E_2}, E_2 \therefore_{\odot} E_1$	$\leq_{\odot}$	$P(E_1) \in \left[P(E_2) \times P\left(\frac{E_1}{E_2}\right), 1 + P(E_2)\left(P\left(\frac{E_1}{E_2}\right) - 1\right)\right]$
Similar to [3]	Denying the Antecedent	$\frac{E_1}{E_2}, -E_2 \therefore_{\odot} -E_1$	$\not\leq_{\odot}$	$P(-E_1) \in \left[\left(1 - P\left(\frac{E_1}{E_2}\right)\right) \times (1 - P(-E_2)), 1 - P\left(\frac{E_1}{E_2}\right) \times (1 - P(-E_2))\right]$
R_2 and R_3	Modus Tollens	$\frac{E_1}{E_2}, -E_1 \therefore_{\odot} -E_2$	$\leq_{\odot}$	$P(-E_2) \in \left[\max\left\{\frac{1-P\left(\frac{E_1}{E_2}\right)-P(-E_1)}{1-P\left(\frac{E_1}{E_2}\right)}, \frac{P\left(\frac{E_1}{E_2}\right)+P(-E_1)-1}{P\left(\frac{E_1}{E_2}\right)}\right\}, 1\right]$
Similar to [3]	Affirming the Consequent	$\frac{E_1}{E_2}, E_1 \therefore_{\odot} E_2$	$\not\leq_{\odot}$	$P(E_2) \in \left[0, \min\left\{\frac{P(E_1)}{P\left(\frac{E_1}{E_2}\right)} \frac{1-P(E_1)}{1-P\left(\frac{E_1}{E_2}\right)}\right\}\right]$
Hailperin (1996: 248)	Hypothetical syllogism	$\frac{B}{E_2}, \frac{E_1}{B} \therefore_{\odot} \frac{E_1}{E_2}$	$\leq_{\odot}$	$P\left(\frac{E_1}{E_2}\right) \in [0,1]$
Hailperin (2011: 34) & R_5	Cut	$\frac{E_2}{E_3}, \frac{E_1}{E_3 \cap E_2} \therefore_{\odot} \frac{E_1}{E_3}$	$\leq_{\odot}$	$P\left(\frac{E_1}{E_3}\right) \in \left[P\left(\frac{E_2}{E_3}\right) \times P\left(\frac{E_1}{E_3 \cap E_2}\right), P\left(\frac{E_2}{E_3}\right) \times P\left(\frac{E_1}{E_3 \cap E_2}\right) + 1 - P\left(\frac{E_2}{E_1}\right)\right]$
Hailperin (1996: 248)	Proofs by cases	$\frac{E_1}{E_2}, \frac{E_1}{-E_2} \therefore_{\odot} E_1$	$\not\leq_{\odot}$	$P(E_1) \in \left[\min\left\{P\left(\frac{E_1}{E_2}\right), P\left(\frac{E_1}{-E_2}\right)\right\}, \max\left\{P\left(\frac{E_1}{E_2}\right), P\left(\frac{E_1}{-E_2}\right)\right\}\right]$
Hailperin (1996: 248)	Reductio ad absurdum	$\frac{E_1}{E_2}, \frac{-E_1}{E_2} \therefore_{\odot} -E_2$	$\leq_{\odot}$	$P(-E_2) \in [0,1]$
Hailperin (2011: 34) & R_5	Cautious monotonicity	$\frac{E_2}{E_3}, \frac{E_1}{E_3} \therefore_{\odot} \frac{E_1}{E_3 \cap E_2}$	$\leq_{\odot}$	$P\left(\frac{E_1}{E_3 \cap E_2}\right) \in \left[\max\left\{0, \frac{P\left(\frac{E_2}{E_3}\right)+P\left(\frac{E_1}{E_3}\right)-1}{P\left(\frac{E_2}{E_3}\right)}\right\}, \min\left\{\frac{P\left(\frac{E_1}{E_3}\right)}{P\left(\frac{E_2}{E_3}\right)}, 1\right\}\right]$
Vidal (2014) & R_5	'Switches'	$\frac{E_1}{E_3 \cap E_2}, \therefore_{\odot} \frac{E_1}{E_3} \cup \frac{E_1}{E_2}$	$\leq_{\odot}$	$P\left(\frac{E_1}{E_3} \cup \frac{E_1}{E_2}\right) \in \left[P\left(\frac{E_1}{E_3 \cap E_2}\right), 1\right]$
[4]	'Not-E_2' to 'if'	$-E_2 \not\therefore_{\odot} \frac{E_1}{E_2}$	$\not\leq_{\odot}$	$P\left(\frac{E_1}{E_2}\right) \in [0,1]$
Similar to [4]	Symmetry	$\frac{E_1}{E_2} \not\therefore_{\odot} \frac{E_2}{E_1}$	$\not\leq_{\odot}$	$P\left(\frac{E_2}{E_1}\right) \in [0,1]$
Hailperin (1996: 249)	Contraposition	$\frac{E_1}{E_2} \not\therefore_{\odot} \frac{-E_2}{-E_1}$	$\not\leq_{\odot}$	$P\left(\frac{-E_2}{-E_1}\right) \in [0,1]$
Similar to [4]	Strengthening	$\frac{E_1}{E_2}, \not\therefore_{\odot} \frac{E_1}{E_2 \cap B}$	$\not\leq_{\odot}$	$P\left(\frac{E_1}{E_2 \cap B}\right) \in [0,1]$

[1] Probability interval are found with water tank analogy of Politzer 2016.
[2] As underlined by Hailperin (1996: 249), $\odot$-validity is not preserved under substitution. e.g. $\frac{A_1 \cap -A_1}{A_1 \cap -A_1}$ is not $\odot$-valid.
[3] When $A = \ominus$, then $\frac{E_1}{E_2} = \odot$ thus $E_1 \therefore_{\odot} \frac{E_1}{E_2}$. However when $A = \ominus$ and $C = \oplus$, $\frac{E_1}{E_2} \cap C = \odot$.
[4] When $\frac{E_1}{E_2} = \odot$, $E_2 \supset E_2 :=: \oplus$.

Table 7: Main Arguments following their $\odot$validity and their probability interval.

4. Conclusion: Primacy of De Finetti's Concepts

Milne (2012) believes that Joseph Schächter was the first author in 1935 to propose Table 1 for indicative conditional *If* E_2, E_1. Recently, Égré, Rossi and Sprenger (2020a) wonder whether the primacy of the truth table might not belong to Hans Reichenbach. Indeed, Reichenbach (1935) 1949: 400 and Reichenbach 1935: 42present a truth table with three values that the author notes '1', '0' and '?' for 'probabilistic implication' $E_2 \ni\!\!- E_1$[27] in the specific 'limiting cases' where

[27] Introduced as early as 1925 (see Reichenbach [1925] 1978: 89-90).

probabilities of E_2 and E_1 are 0 or 1. [28] In the general case, the truth table of the "probabilistic implication" corresponds to a plurivalent logic where the truth values correspond to the numerical values of the degrees of probability. In 1935, de Finetti makes a critical review of this point (see de Finetti 1927-1935: 62-65 and the correspondence 50-61 to which, Reichenbach responds opposing the 2-to-3-valued-logic approach of de Finetti). [29] De Finetti (1936) will specify that this infinite value logic can be reduced to his 2-to-3 valued logic (abandoning Reichenbach's frequentist presuppositions to establish probabilities). It was not until 1941 that Reichenbach presented his three-value quantum logic in a form similar to de Finetti's, but with a different implication (Reichenbach 1944, de Finetti (1970) 1975).

As carefully established by this paper we support that as early as 1928, Bruno de Finetti had expressed the idea of the table for the conditional and has already conceptualized the whole logic of tri-events.[30]

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[28] The general case being formulated by an inequality close to the interval of the "introduction of the conditional" (see Table 7): $P\left(\frac{E_1}{E_2}\right)$ (noted "u") $\in \left[\frac{(P(E_1)+P(E_2)-1)}{P(E_2)}, \frac{P(E_1)}{P(E_2)}\right]$. The three values "1", "0" and "?" represent respectively the values of u for $P(E_1) = P(E_2) = 1$, $P(E_1) = 0$ and $P(E_2) = 1$, and $P(E_2) = 0$.

[29] The text can also be found at http://www.brunodefinetti.it/Opere/Rec%20B.de%20Finetti-Hans%20Reichenbach.pdf.

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