Bypassing Lewis' Triviality Results. A Kripke-Style Partial Semantics for Compounds of Adams' Conditionals

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Abstract

According to Lewis' Triviality Results (LTR), conditionals cannot satisfy the equation (E) $\mathbf{P}(C\ if\ A) = \mathbf{P}(C \mid A)$, except in trivial cases. Ernst Adams (1975), however, provided a probabilistic semantics for the so-called *simple conditionals* that also satisfies equation (E) and provides a probabilistic counterpart of logical consequence (called *p*-entailment). Adams' probabilistic semantics is coextensive to Stalnaker-Thomason's (1970) and Lewis' (1973) semantics as far as simple conditionals are concerned. A theorem, proved in McGee 1981, shows that no truth-functional many-valued logic allows a relation of logical consequence coextensive with Adams' *p*-entailment.

This paper presents a modified modal (Kripke-style) version of de Finetti's semantics that escapes McGee's result and provides a general truth-conditional semantics for indicative conditionals. It agrees with Adams' logic and is not affected by LTR. The new framework encompasses and extends Adams' probabilistic semantics (APS) to compounds of conditionals. A generalised set of axioms for probability over the set of tri-events is provided, which coincide with the standard axioms over the set of the two-valued ordinary sentences.

Keywords: Conditionals, Probability logic, de Finetti, Tri-events, Adams' logic, Stalnaker's thesis, Partial logic, Lewis' triviality results, Ramsey test.

1. Introduction: de Finetti's Tri-events

According to the so-called Stalnaker's Thesis,[1] the probability $\mathbf{P}(p \Rightarrow q)$ of an indicative conditional "if p then q" equals the probability $\mathbf{P}(q \mid p)$ of the consequent given the antecedent. However, according to so-called Lewis' Triviality Results

[1] I use the expression 'Stalnaker's Thesis' because Stalnaker 1970 prompted its consideration among philosophers, and it is often so-called in the literature on conditionals. As a matter of fact, the idea goes back to Ramsey (1928) 1990 and also to de Finetti (1934) 2006.

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(Lewis 1976), hereafter abbreviated by 'LTR', there is no connective "$\Rightarrow$" that satisfies, in general, the equation $\mathbf{P}(p \Rightarrow q) = \mathbf{P}(q \mid p)$, except in trivial cases. These results seem to support either the view, maintained by several scholars, according to which conditionals lack in general truth conditions or the view that the equation is incorrect, at least concerning non-simple conditionals.

LTR (on all its versions, including Hájek 1989, that appears to be one version that limits the set of assumptions to the non-dispensable ones) rest on their premises. Among them, two appear to be crucial:

1. Conditional sentences express two-valued propositions, so that n $(0 < n < \omega)$ conditional sentences generate a Boolean algebra with at most 2^n elements (up to logical equivalence);

2. the laws of finite probability hold for conditionals as they are. No modification is required.

Condition (2) is natural in the presence of the condition (1), while denial of condition (1) not necessarily entails the denial of condition (2). What if we drop both these two premises? Bruno de Finetti ([1934] 2006, [1935] 1995) proposed a new framework, where (a) conditionals (called *tri-events*) have partial truth conditions and (b) probabilities are defined over a lattice which, if genuine tri-events are involved, is not a Boolean algebra. [2] In de Finetti's approach, the probability of ordinary sentences follows the common laws of finite probability, while the probability of tri-events follows slightly more general laws, always consistently satisfying the equation $\mathbf{P}(p \Rightarrow q) = \mathbf{P}(q \mid p)$. Moreover, when p and q are ordinary two-valued sentences, it holds that if $\mathbf{P}(p) > 0$, $\mathbf{P}(p \Rightarrow q) = \frac{\mathbf{P}(p \wedge q)}{\mathbf{P}(p)}$, so that in the case of simple conditionals (that is in the case of those conditionals in which both the antecedent and the consequent express ordinary two-valued propositions), conditional probability follows the usual ratio formula.

De Finetti did not define the relation of logical consequence between tri-events. By contrast, Ernest Adams (1975) developed a probabilistic semantics for simple conditional sentences with no impossible antecedent (hereafter abbreviated by 'APS'). He defined for such conditionals a logical consequence relation in probabilistic terms (called p-entailment). Adams p-entailment coincides with the standard logical consequence when only ordinary sentences are involved. Moreover, p-entailment is in excellent agreement with intuitions in most cases. It turns out (Adams 1977, see also Gibbard 1981) that, as far as simple conditionals with no impossible antecedent are concerned, the logical consequence relations defined respectively in the Stalnaker-Thomason semantics (1970) and the Lewis' semantics (1973) are coextensive to p-entailment.

According to Adams, conditionals lack truth conditions except in the case in which they come down to ordinary sentences. For Adams, p-entailment belongs

[2] Recently, J. Baratgin (2021) showed that de Finetti developed in great detail in his unpublished manuscripts held at the University of Pittsburgh (1927-1935) his theory of tri-events as early as 1928. For very recent discussions and development of definettian logics with respect to its three-valued alternatives see Égré, Rossi and Sprenger 2021a, 2021b, Lassiter and Baratgin 2021. One can find a previous account of definettian logics in Milne 2004.

to a probabilistic semantics in which the correctness of an inference cannot be presented as some genuine preservation property concerning the truth conditions of the premises and the conclusion. However, to say that *p*-entailment preserves high probability is somewhat misleading: when the number of the premises is sufficiently large, the probability of each premise can be very high (short of full certainty),[3] while the probability of a *p*-entailed conclusion very low. Moreover, as Kleiter (2018) points out, there are inferences (like affirming the consequent) that are not *p*-valid, while "allow to constrain the probabilities of conclusions in the same way as the modus ponens or the modus tollens", and are such that, from the viewpoint of a probabilistic logic without truth conditions, should not be considered as fallacious. To vindicate Adams' probability logic we need a parallel truth-conditional logic. And extending Adams' probability logic to compounds of conditionals requires a new truth-conditional semantics.

Is it possible to equip de Finetti's logic of tri-events with a logical consequence relation that agrees with Adams *p*-entailment? Due to a theorem by McGee (1981), this is not the case (on this point see also Adams 1995, Schulz 2009).

A closer analysis of de Finetti's logic shows that, concerning simple conditionals with no impossible antecedents, it differs from Adams' logic because every sentence of the form $\varphi \Rightarrow \varphi$, while is probabilistically valid in Adams' logic, is not valid in de Finetti's logic. Indeed, while it may not be false, it may well not be true.

This consideration motivates the present effort to provide a new semantics for tri-events such that every sentence of the form $\varphi \Rightarrow \varphi$ turns out to be true whenever φ does express a proposition that may be true. Since truth depends here on a modal condition, to express this fact by a schema, we need a language with modal operators.

This paper presents such a semantical account in a Kripke-style manner. We will define probability axioms and probability functions that satisfy, in a general way, the equation $\mathbf{P}(p \Rightarrow q) = \mathbf{P}(q \mid p)$. We can define both notions of *p*-entailment and truth-conditional logical consequence in this theory. It turns out that our *p*-entailment encompasses Adams *p*-entailment since it coincides with it concerning simple conditionals. Moreover, a relation of logical consequence that is coextensive with general *p*-entailment is definable in truth-conditional terms. Like Adams

[3] McGee (1994) has refined Adams' *p*-entailment by the so-called Popper probability functions. McGee defines the notion of *strict entailment* (renamed in Adams 1998: 152, *weak validity* after recognition that it is not stricter but weaker than *p*-entailment), that preserves probability 1. McGee approach has the valuable advantage that strict entailment is compact so that it needs no restriction to finite sets of premises. Despite that, we stick to the original Adams' 1975 approach. Following Adams 1975: 49-50, we require that in probabilistic arguments, probability assignments are *proper* for $(\psi \mid \varphi)$ if $\mathbf{P}(\varphi) > 0$ and that only proper assignments occur in probabilistic arguments. Our truth-conditional semantics fits this notion of *p*-entailment. The main reason for this choice is that when $\mathbf{P}(\varphi) = 0$ and φ is a contingent sentence, the probability that $(\psi \mid \varphi)$ lacks a truth-value is 1 in the present approach. Since we consider probability as probability of truth, it does not make sense (at least as far as inferential reasoning is concerned) to assign a definite probability value to sentences that, with probability 1, lack a truth-value. Further considerations will be made in section 6.

probabilistic semantics, we confine the present truth-conditional semantics to the so-called *indicative conditionals*.[4]

The present theory provides a new semantical account for indicative conditionals, which is alternative to other semantics (like Stalnaker-Thomason's one) as far as compounds of indicative conditionals are concerned. I discuss the problem whether this semantics is better suited in dealing with compounds of conditionals in the final part of the paper.

While de Finetti in (1935) 1995 presented his logic as a three-valued logic, actually he meant it as a *partial* logic. Indeed, he meant the third value out of true and false not as *undetermined* (that is as unknown or unknowable), but as *null*, that is a genuine truth-value gap. De Finetti maintained throughout his life (see de Finetti [1979] 2008: 169) that we may interpret '|' in a probability statement of the form $\mathbf{P}(q \mid p)$ as a logical connective and that a sentence of the form $(q \mid p)$ may be interpreted as 'q supposing that p', according to the following truth-table:

Conditioning
$q \mid p$

		q		
		1	0	u
	1	1	0	u
p	0	u	u	u
	u	u	u	u

When p and q are ordinary events, a tri-event $(q \mid p)$ is considered true if both p and q are true, false if p is true while q is false and null (i.e. neither true nor false) if p is false.

De Finetti did not aim at solving philosophical problems about conditionals, as logicians and philosophers of language today understand them. Instead, his problem was to extend to conditional probability a well-known fact that holds for absolute (finite) probability: that the laws of probability force the probability of a proposition p to be 1 (or 0) if and only if p is logically true (or logically false). Since the laws of finite probability force also, in special cases, *conditional* probabilities or compound of conditional probabilities to be 0 or 1, the problem arises whether also in this case these extreme probability values correspond to truth-values. This approach would require an extension of Boolean logic to include conditional events. This requirement explains why de Finetti named his theory of tri-events 'the logic of probability'.

For example, the following equation holds for conditional probability:

(1) $\mathbf{P}(q \mid p) + \mathbf{P}(\neg q \mid p) = 1$

provided p is possibly true.[5] In the case of absolute probability the equation $\mathbf{P}(q) + \mathbf{P}(\neg q) = 1$ is linked to the logical fact that either q or $\neg q$ is true. Can we interpret

[4] In my paper Mura 2016, I have outlined an extension of the present theory, covering counterfactual conditionals as well. This extension will not be considered here.

[5] de Finetti 1936 anticipates (without providing an axiomatic treatment) the idea that conditional probability may be defined even when it is conditional on a 0-probability event.

equation (1) in a way similarly linked to the truth conditions of $\mathbf{P}(q \mid p)$ and $\mathbf{P}(\neg q \mid p)$? De Finetti theory of tri-events aims at providing a positive answer to questions like this. As we shall see, the original theory proposed by de Finetti does not allow this conclusion, while the modified modal theory we propose does.

Another route to tri-events comes from consideration of bets. If a bet is on a two-valued event E, the bettor wins the bet if E obtains, that is, if the sentence that expresses it is true and is lost if it is false. This remark suggests that bets are at the intersection of sentence logic and probability. De Finetti models conditional probabilities on an event E given another event H by conditional bets, that is, bets that the bettor wins if both H and E occur, that the bettor loses if H occurs, but E does not, and that is called for if H does not occur. De Finetti developed his theory of probability in terms of coherence in the betting behaviour. In such a framework, he identifies the probability of an event E with that betting quotient which the bettor considers 'fair', that is at which the bettor is indifferent between betting on E or betting against E.[6] As far as conditional bets are concerned, if winning is associated with truth and losing with falsehood, calling for the bet is associated with a truth-value gap. In this way, a bet on an event E conditional to another event H is seen as a bet on a *conditional event* $(E \mid H)$ which is expressed by a conditional sentence, meant in a suppositional way: "E if H" or "E on the supposition that H". This conditional event expresses a true sentence if both H and E occur, is false if H is true while E is false, and is 'null' if H does not occur. So the truth-conditional semantics of tri-events is seen as the semantical counterpart of the logic of probability, including conditional probability, in terms of bets.

There is also a third Definettian route to tri-events. According to de Finetti, the probability of a proposition is the expectation of its truth-value (he used the bold letter '**P**' to designate both probability and expectation, renamed 'prevision'). How may we apply this view to conditional probability? The answer resorts to the idea of *conditional expectation*. The conditional expectation is a well-known notion of probability theory. Where X is a random number and p an event, it holds that $\mathbf{P}_p(X) = \frac{\mathbf{P}(X \times p)}{\mathbf{P}(p)}$ if $\mathbf{P}(p) > 0$.

When X is, in turn, an event (say q), we have $\mathbf{P}_p(q) = \frac{\mathbf{P}(q \times p)}{\mathbf{P}(p)} = \mathbf{P}(q \mid p)$.

Although, as we have already said, de Finetti had not the intent of solving a problem about conditional sentences, the theory of tri-events actually provides an *ante litteram* solution to a problem about conditionals with which more recently philosophers have struggled: to find a connective '$\Rightarrow$' such that $\mathbf{P}(p \Rightarrow q) = \mathbf{P}(q \mid p)$. Indeed, if '|' is a connective and $\mathbf{P}(q \mid p)$ equals the conditional probability of q given p, the '|' is just the wanted connective.

As already pointed out, this solution may seem at odds with LRT. However, this is not the case. For, LRT, even in the strongest form proved by Hájek (1989), assumes that conditionals are two-valued propositions, a premise that in the de Finetti's theory does not hold.

[6] If, as de Finetti does, one assumes that the final goal of the bettor is maximising expected gain (supposed linear in utility), one may equivalently characterise the fairness of bets as indifference between betting on E and abstaining from betting. The characterisation in the text does not require that rule of maximisation of expected gain is satisfied (see Mura 1994).

The standard axioms of probability apply to Boolean algebras. Defining probability on other structures requires some adjustment (as happens, for example, in Quantum Mechanics). Since, algebraically, the field of tri-events is a lattice but not, in general, a Boolean algebra, it is quite natural that the laws of probability must be adjusted accordingly. Such a lattice contains as a sub-lattice the Boolean algebra of two-valued events. Of course, as far as two-valued ordinary events are concerned, the usual laws of probability must be satisfied. Since ordinary events are special cases of tri-events, there is no harm in generalised probability laws that turn out to coincide with the usual laws when only ordinary events are involved. In section 6.1 we provide a new set of probability axioms that apply in a general way to tri-events and are such that reduce to the standard axioms of conditional probability when only ordinary events are involved.

As remarked above, de Finetti's presents his theory as a three-valued logic, where the third value beyond true and false is meant as a truth-value gap. Truth-tables define the semantics of connectives. De Finetti takes negation, disjunction and conjunction from Łukasiewicz (1930) 1987 three-valued logic. The implication is the same as in Kleene 1938 strong three-valued logic (though de Finetti introduced it before Kleene). The qualifying connective is conditioning, whose truth-table I have reported above. Along with these three-valued connectives, de Finetti's semantics contains, like in Bochvar (1937) 1981, two unary connectives that transform a three-valued event in a two-valued one. We define them by the following truth-table:[7]

	Thesis	Hypothesis
p	$\uparrow p$	$\updownarrow p$
1	1	1
0	0	1
u	0	0

These connectives are important in many respects.[8] First, they allow to show that every sentence that expresses a compound of tri-events has the same truth conditions of a sentence that expresses a *simple tri-event*, that is a sentence of the form $(\varphi \mid \psi)$ where both φ and ψ are two-valued sentences. In fact, in general it holds the equivalence $\chi = (\uparrow \chi \mid \updownarrow \chi)$. Second, they allow the algebra of tri-events to contain the Boolean algebra of binary events (whose elements are those tri-events φ such that $\varphi = \uparrow \varphi$). As far as probability is concerned, Bayes Theorem in the standard form $\mathbf{P}(\psi \mid \varphi) = \frac{\mathbf{P}(\psi \wedge \varphi)}{\mathbf{P}(\varphi)}$ (provided $\mathbf{P}(\varphi) > 0$) does not hold in general for tri-events, albeit it holds with two-valued tri-event. However, the more general formula $\mathbf{P}(\varphi) = \frac{\mathbf{P}(\uparrow \varphi)}{\mathbf{P}(\updownarrow \varphi)}$ (provided $\mathbf{P}(\updownarrow \varphi) > 0$) is always satisfied.

[7] The symbolism used here differs from de Finetti's symbolism. He used 'T' (for 'Thesis') instead of our '↑' and 'H' (for 'Hypothesis') instead of our '↕'.

[8] In what follows I will call *events* or *propositions* the equivalence classes defined on the set of sentences induced by the relation of having the same truth conditions.

Is de Finetti's semantics compatible with Adams' probabilistic semantics? Can we define a logical consequence relation coextensive with Adams *p*-entailment? The answer is in the negative. This result follows from a theorem by Mc Gee (1981), according to which no *m*-valued truth-functional logic, even if equipped with modal operators, allows the definition of a relation of logical consequence preserving designated values and coextensive to *p*-entailment. As already pointed out, in the case of de Finetti's semantics, this is connected to the fact that tri-events of the form $(\varphi \mid \varphi)$ are *not* valid formulas. Tri-events of the form $(\varphi \mid \varphi)$ are quasi-tautologies, that is they may be either true or null. We cannot convene to equate quasi-tautologies to tautologies in a general way because those tri-events that are necessarily null (call it *singular tri-events*, see below Definition 2.6) are quasi-tautologies and are also truth-functionally equivalent to their negation, so that, in such a case, the negation of a tautology would be a tautology, which would amount to an inconsistency.

2. The Modal Semantics of Tri-events: Preliminary Explanations

The modal semantics proposed here aims at defining truth-conditional semantics for tri-events. Such semantics includes a logical consequence relation that generalises the common logical consequence for two-valued ordinary sentences and is coextensive with Adams *p*–entailment (extended to the lattice of tri-events). According to this semantics, a sentence of the form $(\varphi \mid \varphi)$ is not true only when φ is either singular or logically false. As anticipated above, we may express this logical fact by the schema $(\Diamond \updownarrow \varphi \;\rightarrow \uparrow (\varphi \mid \varphi))$ (where '$\rightarrow$' is a special material conditional, whose truth conditions will be specified below (see page 304).

The basic idea consists of defining the truth conditions so that no sentence is not false at every world and true at some but not at all worlds. In the same vein, no sentence fails to be true at every world and false at some world that is not false at every world. Underlying this tenet is the idea that, in the context of partial logic, a sentence φ must be considered as valid (unsatisfiable) if and only the two following conditions are satisfied: (a) φ is true (false) at all possible worlds in which it has a truth value and (b) there is a possible world at which φ is true (false). Clause (a) is required if we take seriously the idea that a sentence that is neither true nor false lacks a truth-value rather than bearing a third genuine truth-value. Clause (b) is required because sentences that cannot be true (false) would be considered as necessarily true (false), which seems absurd. This notion of validity (unsatisfiability) reduces to the standard notion when φ is a two-valued sentence. The truth conditions of atomic sentences are re-valuated in such a way that those atomic sentences that satisfy clauses (a) and (b) are considered as true (false) at every world. [9] Moreover, the truth conditions of some connectives are warped in such a way that molecular sentences satisfying clauses (a) and (b) turn out to be

[9] This re-valuation resembles van Fraassen's *supervaluation* (1966). However, it differs in several respects and affects the truth conditions of molecular sentences in a general way. Elsewhere (Mura 2009) I have called my re-evaluation of sentences 'hypervaluations.' In the present Kripke-style framework, however, no formal definition of hypervaluations is required.

true (false) at every world. Connectives so modified have a modal import, and modal operators are definable by them.

So, by this semantics, sentences are exhaustively so classified concerning a given model:

(a) Necessarily true (true at every world);
(b) Necessarily false (false at every world);
(c) Singular (neither true nor false at every word);
(d) Contingent (true at some world and false at some world).

Among the contingent sentences another distinction is relevant: the distinction between those contingent sentences that at every world are either true or false (we call *two-valued* such sentences) and those contingent sentences that lack a truth-value at one or more worlds. The class of two-valued contingent sentences joined with the class of necessarily true sentences and with the class of necessarily false sentences is closed under all connectives except '|' conditioning connective. So the present semantics extends the standard sentential modal S5 semantics rather than replacing it. Atomic sentences may belong to each of the categories above. If they are contingent, they can be two-valued or not, depending on the model.

A special feature of this semantics is that *some of the binary connectives carry a modal import*, i.e. their truth conditions refer to the set of possible worlds. As a result, we may define modal operators in terms of the other connectives. However, we prefer to put explicitly the truth conditions of modal operators. By contrast, all unary connectives are truth-functional. We stress that the modal aspects of binary connectives do not affect completely independent sets of non-singular sentences[10] so that in many examples discussed in the literature the truth conditions of these connectives coincide with the original de Finetti's truth conditions. The main modification of the proposed semantics consists in considering true (false) at a given world a sentence which, according to de Finetti's semantics, would be null whenever it is not false (true) at every world and true (false) at some world.

2.1 The Modal Semantics in Detail

2.1.1 The Language $\mathscr{L}$

Constants: $\bot$, $\neg$. $\uparrow$, $\vee$, $\wedge$, $|$, $\rightarrow$, $\Diamond$
Atomic sentences: $\mathbb{P}_0$, $\mathbb{P}_1$,...

Definition 2.1 (Sentences)

The string φ of symbols of $\mathscr{L}$ is a sentence iff at least one of the following conditions is satisfied (where the metalinguistic symbol '$\approx$' means 'has the same form as'):

(a) φ is an atomic sentence;

[10] A set of non-singular sentences U is *completely independent* iff for arbitrary disjoint finite subsets $\{u_1,\dots,u_n\}$, and $\{v_1,\dots,v_m\}$ of U, the set $\{u_1,\dots,u_n, \neg v_1,\dots,\neg v_m\}$ is satisfiable (see definitions 2.7 and 2.8 below).

(b) φ = '$\bot$';

(c) either $\varphi \approx \neg\psi$ or $\varphi \approx\uparrow \psi$ or $\varphi \approx \Diamond\psi$, where ψ is a sentence;

(d) either $\varphi \approx (\psi \vee \chi)$ or $(\psi \wedge \chi)$ or $\varphi \approx (\psi \to \chi)$ where both ψ and χ are sentences;

(e) $\varphi \approx (\psi \mid \chi)$ where both ψ and χ are sentences.

Definition 2.2 (Metalinguistically defined logical constants)

$$\top \stackrel{\text{def}}{=} (\bot \to \bot)$$

$$\natural \stackrel{\text{def}}{=} (\bot \mid \bot)$$

$$(\varphi \leftrightarrow \psi) \stackrel{\text{def}}{=} ((\varphi \to \psi) \wedge (\psi \to \varphi))$$

$$\downarrow \varphi \stackrel{\text{def}}{=} (\neg\uparrow\varphi \wedge \neg\uparrow\neg\varphi)$$

$$\updownarrow \varphi \stackrel{\text{def}}{=} (\uparrow\varphi \vee \uparrow\neg\varphi)$$

$$\Box\varphi \stackrel{\text{def}}{=} \neg\Diamond\neg\varphi$$

The set of the primitive logical constants of $\mathscr{L}$ is redundant. For example, according to the semantic definition below, $\Diamond$ is definable, since the following equivalence holds at every world in every model: $\Diamond\varphi \leftrightarrow ((\top \to (\varphi \mid \varphi)) \mid \uparrow(\varphi \vee \neg\varphi))$. However, in a semantical context, this redundancy presents several well-known advantages. The following definition is adapted from Chellas 1980. We symbolise that a sentence φ is true (false) at world α in the model $\mathcal{M}$ by $\models_\alpha^{\mathcal{M}} \varphi$ ($\overline{\models}_\alpha^{\mathcal{M}} \varphi$).

Definition 2.3 (Models)

A model is a triplet $\mathcal{M} = (W, P, Q)$, where W is a set, whose elements are called 'worlds', P and Q are functions of natural numbers such that for each number n, P_n and Q_n are subsets of W (i.e. $P : \mathbb{N} \to \wp(W)$; $Q : \mathbb{N} \to \wp(W)$), and for each n, Q_n is a subset of P_n.

Definition 2.4 (Truth-conditions at a possible world α in the model $\mathcal{M}$)

Let α be a world in a model $\mathcal{M} = (W, P, Q)$.

1.

(a) $\mathbb{P}_n$ is true at α (where $n \in \mathbb{N}$): $\models_\alpha^{\mathcal{M}} \mathbb{P}_n$ iff either $\alpha \in Q_n$ or both the following conditions are satisfied:

(i) $P_n = Q_n$;

(ii) $Q_n \neq \emptyset$.

(b) $\mathbb{P}_n$ is false at α: $\overline{\models}_\alpha^{\mathcal{M}} \mathbb{P}_n$ iff either $\alpha \in P_n - Q_n$ or both the following conditions are satisfied:

(i) $Q_n = \emptyset$
(ii) $P_n \neq \emptyset$.

COMMENT. Atomic sentences may be, at any world, either true or false or neither true nor false. Every atomic sentence $\mathbb{P}_n$ has associated two subsets of the set of worlds, the set P_n and the set Q_n. P_n is *a* set of worlds in which the atomic sentence $\mathbb{P}_n$ is either true or false, Q_n is *a* set of worlds in which the atomic sentence $\mathbb{P}_n$ is true (where $Q_n \subseteq P_n$). If P_n and Q_n would contain as elements, respectively, *all* the true atomic sentences and *all* those atomic sentences that are either true or false, then they, taken together, would define a common valuation V of atomic sentences. In our semantics, however, sentences that do not belong to Q_n (so that by V they are neither true nor false) are re-valuated in such a way that even if $\alpha \notin P_n$, $\mathbb{P}_n$ may be true or false. In such a case, $\mathbb{P}_n$ is true at α if $\mathbb{P}_n$ is true at some world $\in Q_n$ and false at no world $\in P_n$, and $\mathbb{P}_n$ is false at α if $\mathbb{P}_n$ is false at some world $\in P_n$ and true at no world $\in Q_n$. The truth conditions of atomic sentences are well-defined: at every world, a sentence is either true or false or neither true nor false. It should be stressed that the truth conditions of atomic sentences at a given world have a modal import since they may depend on conditions about the totality of worlds. The present approach differs from the common treatment of conditional events that typically moves from atomic two-valued sentences (or events). The two approaches are, after all, equivalent and differ only in technical details. However, considering atomic sentences that may lack a truth-value is in a better agreement with the view that two-valued sentences are just a special case and not the basic sentences upon which tri-events are built.

2.
   (a) $\underline{\mathrm{N}}\mathrm{ot} \models^{\mathcal{M}}_{\alpha} \bot$;
   (b) $\overline{\models}^{\mathcal{M}}_{\alpha} \bot$.

COMMENT. '$\bot$' is false at every world.

3.
   (a) $\underline{\models}^{\mathcal{M}}_{\alpha} \neg\varphi$ iff $\overline{\models}^{\mathcal{M}}_{\alpha} \varphi$;
   (b) $\overline{\models}^{\mathcal{M}}_{\alpha} \neg\varphi$ iff $\models^{\mathcal{M}}_{\alpha} \varphi$.

COMMENT. $\neg\varphi$ is true at those worlds at which φ is false and false at those worlds at which φ is true. The semantics of '$\neg$' carries no modal import beyond the possible modal import of the involved atomic sentences, being truth-functional.

4.
   (a) $\underline{\models}^{\mathcal{M}}_{\alpha} \uparrow\varphi$ iff $\models^{\mathcal{M}}_{\alpha} \varphi$;
   (b) $\overline{\models}^{\mathcal{M}}_{\alpha} \uparrow\varphi$ iff not $\models^{\mathcal{M}}_{\alpha} \varphi$.

COMMENT. $\uparrow\varphi$ is true at a world α iff φ is true at α and false at α otherwise. So '$\uparrow\varphi$' is a two-valued sentence. The semantics of '$\uparrow$' carries no modal import, being truth-functional.

5.
(a) $\models^{\mathcal{M}}_{\alpha} (\varphi \vee \psi)$ iff either $\models^{\mathcal{M}}_{\alpha} \varphi$ or $\models^{\mathcal{M}}_{\alpha} \psi$ or all the following conditions are satisfied:
   (i) for no β in $\mathcal{M}$ it holds that $\overline{\models}^{\mathcal{M}}_{\beta} \varphi$ and $\overline{\models}^{\mathcal{M}}_{\beta} \psi$;
   (ii) there exists β in $\mathcal{M}$ such that either $\models^{\mathcal{M}}_{\beta} \varphi$ or $\models^{\mathcal{M}}_{\beta} \psi$.

(b) $\overline{\models}^{\mathcal{M}}_{\alpha} (\varphi \vee \psi)$ iff both $\overline{\models}^{\mathcal{M}}_{\alpha} \varphi$ and $\overline{\models}^{\mathcal{M}}_{\alpha} \psi$ or all the following conditions are satisfied:
   (i) for no β in $\mathcal{M}$ it holds that $\models^{\mathcal{M}}_{\beta} \varphi$ or $\models^{\mathcal{M}}_{\beta} \psi$;
   (ii) there exists β in $\mathcal{M}$ such that both $\overline{\models}^{\mathcal{M}}_{\beta} \varphi$ and $\overline{\models}^{\mathcal{M}}_{\alpha} \psi$.

COMMENT. $(\varphi \vee \psi)$ is a modified version of Łukasiewicz three-valued disjunction (where the third truth-value out of true and false is interpreted as a truth-value gap). More exactly, in a model $\mathcal{M}$, $(\varphi \vee \psi)$ is true at world α if either (a) φ or ψ is true at α or (b) both φ and ψ are not false at every world and either φ or ψ is true at some world. $(\varphi \vee \psi)$ is false at α if either (a) neither φ nor ψ are true at α or (b) $(\varphi \vee \psi)$ is false at α and there is no world at which either φ or ψ is true and there is a world at which both φ and ψ are false. Notice that the disjunction connective typically has a modal import, since its truth conditions may depend not only on the truth-values of the involved sentences but also on the truth-valued across the totality of worlds.

6.
(a) $\models^{\mathcal{M}}_{\alpha} (\varphi \wedge \psi)$ iff either both $\models^{\mathcal{M}}_{\alpha} \varphi$ and $\models^{\mathcal{M}}_{\alpha} \psi$ or both the following conditions are satisfied:
   (i) For no world β in $\mathcal{M}$ it holds that either $\overline{\models}^{\mathcal{M}}_{\alpha} \varphi$ or $\overline{\models}^{\mathcal{M}}_{\alpha} \psi$
   (ii) There is a world β in $\mathcal{M}$ such that $\models^{\mathcal{M}}_{\alpha} \varphi$ and $\models^{\mathcal{M}}_{\alpha} \psi$

(b) $\overline{\models}^{\mathcal{M}}_{\alpha} (\varphi \wedge \psi)$ iff either $\overline{\models}^{\mathcal{M}}_{\alpha} \varphi$ or $\overline{\models}^{\mathcal{M}}_{\alpha} \psi$ or all the following conditions are satisfied:
   (i) For no β in $\mathcal{M}$ it holds that $\models^{\mathcal{M}}_{\beta} \varphi$ and $\models^{\mathcal{M}}_{\beta} \psi$;
   (ii) at least one of the following conditions is satisfied:
      (A) there exists γ in $\mathcal{M}$ such that $\overline{\models}^{\mathcal{M}}_{\gamma} \varphi$
      (B) there exists γ in $\mathcal{M}$ such that $\overline{\models}^{\mathcal{M}}_{\gamma} \psi$.

COMMENT. $(\varphi \wedge \psi)$ is a modified version of Łukasiewicz three-valued conjunction (where the third truth-value out of true and false is interpreted as a truth-value gap). More exactly, in a model $\mathcal{M}$, $(\varphi \wedge \psi)$ is true at a world α if both φ and ψ are true at α. $(\varphi \wedge \psi)$ is also true at α if at no world either φ or ψ are false, and there is a world at which both φ and ψ are true. $(\varphi \wedge \psi)$ is false at α if either φ or ψ is false at α or φ, and ψ are not both true at any world and either there is a world at which φ is false, or there is a world at which ψ is false. From theorem 13 below it follows that $\models^{\mathcal{M}}_{\alpha} (\varphi \wedge \psi)$ iff $\models^{\mathcal{M}}_{\alpha} \varphi$ and $\models^{\mathcal{M}}_{\alpha} \psi$. Notice that the conjunction connective, like disjunction, has a modal import since its truth conditions depend not only on the truth-values of the involved sentences but also on the truth-valued across the totality of worlds.

7.
(a) $\models^{\mathcal{M}}_{\alpha} (\varphi \mid \psi)$ iff either $\models^{\mathcal{M}}_{\alpha} \varphi$ and $\models^{\mathcal{M}}_{\alpha} \psi$ or the two following conditions are satisfied:
   (i) there exists β in $\mathcal{M}$ such that $\models^{\mathcal{M}}_{\beta} \varphi$ and $\models^{\mathcal{M}}_{\beta} \psi$;
   (ii) for no β in $\mathcal{M}$ it holds that $\models^{\mathcal{M}}_{\beta} \psi$ and $\overline{\models}^{\mathcal{M}}_{\beta} \varphi$.

(b) $\overline{\models}^{\mathcal{M}}_{\alpha} (\varphi \mid \psi)$ iff either both $\models^{\mathcal{M}}_{\alpha} \psi$ and $\overline{\models}^{\mathcal{M}}_{\alpha} \varphi$, or the two following conditions are satisfied:
   (A) there exists β in $\mathcal{M}$ such that $\models^{\mathcal{M}}_{\beta} \psi$ and $\overline{\models}^{\mathcal{M}}_{\beta} \varphi$;
   (B) for no β in $\mathcal{M}$ it holds that $\models^{\mathcal{M}}_{\beta} \varphi$ and $\models^{\mathcal{M}}_{\beta} \psi$.

COMMENT. $(\varphi \mid \psi)$ is a modified version of de Finetti's conditioning. In a model $\mathcal{M}$, $(\varphi \mid \psi)$ is true at α if both φ and ψ are true at α. Moreover, $(\varphi \mid \psi)$ is true at α if at no world β, ψ is true at β while φ is false at β *and* there exists a world at which both φ and ψ are true. $(\varphi \mid \psi)$ is false at α if ψ is true at α while φ is false at α. Moreover, $(\varphi \mid \psi)$ is false at α if there is no world β such that ψ is true at β and φ is false at β *and* there is no world at which both φ and ψ are true. Notice that also the conditioning connective has a modal import, since its truth conditions depend not only on the truth-values of the involved sentences, but also on the truth-valued across the totality of worlds.

8.
(a) $\models^{\mathcal{M}}_{\alpha} (\varphi \rightarrow \psi)$ iff at least one of the following conditions are satisfied:
   (i) $\overline{\models}^{\mathcal{M}}_{\alpha} \varphi$;
   (ii) $\models^{\mathcal{M}}_{\alpha} \psi$;
   (iii) not $\models^{\mathcal{M}}_{\alpha} \varphi$ and not $\overline{\models}^{\mathcal{M}}_{\alpha} \psi$.

(b) $\overline{\models}^{\mathcal{M}}_{\alpha} (\varphi \rightarrow \psi)$ iff at least one of the following conditions are satisfied:
   (i) $\models^{\mathcal{M}}_{\alpha} \varphi$ and not $\models^{\mathcal{M}}_{\alpha} \psi$;
   (ii) not $\overline{\models}^{\mathcal{M}}_{\alpha} \varphi$ and $\overline{\models}^{\mathcal{M}}_{\alpha} \psi$.

COMMENT. $(\varphi \rightarrow \psi)$ is a special material conditional. $(\varphi \rightarrow \psi)$, at given world α in a model $\mathcal{M}$, is either true or false, so it is a two-valued sentence. $(\varphi \rightarrow \psi)$ is true at α if the antecedent φ is false at α or the consequent ψ is true at α. In addition, $(\varphi \rightarrow \psi)$ is true at α if φ is not true at α and ψ is not false at α. On the other hand, $(\varphi \rightarrow \psi)$ is false at α if φ is true at α and ψ is not true at α. Moreover, $(\varphi \rightarrow \psi)$ is false at α if φ is not false at α while ψ is false at α. The semantics of '$\rightarrow$' carries *no* modal import, being truth-functional.

9.
(a) $\models^{\mathcal{M}}_{\alpha} \Diamond\varphi$ iff there exists β in $\mathcal{M}$ such that $\models^{\mathcal{M}}_{\alpha} \varphi$;
(b) $\overline{\models}^{\mathcal{M}}_{\alpha} \Diamond\varphi$ iff for no β in $\mathcal{M}$ it holds that $\models^{\mathcal{M}}_{\beta} \varphi$ and there exists β in $\mathcal{M}$ such that $\overline{\models}^{\mathcal{M}}_{\alpha} \varphi$.

COMMENT. $\Diamond\varphi$ is true iff φ is true at some world. $\Diamond\varphi$ is false at α iff it is true at no world and false at some world. So, $\Diamond\varphi$ is neither true nor false only if φ is neither true nor false at every world. Note that $\Diamond\varphi$ does not say that φ is possibly true but conditionally says that φ is possibly true supposing that it is not the case that φ is singular (that is neither true nor false at every world). Indeed, if φ is singular (see 2.6 below), neither condition (a) nor condition (b) is satisfied at any world, so that $\Diamond\varphi$ is neither true nor false at every world.

Theorem 2.1 (Truth-conditions for defined connectives)

Let α be a world in a model $\mathcal{M} = (W, P, Q)$. The truth conditions for non–primitive logical constants are the following (proofs omitted):

1

- (a) $\models_\alpha^\mathcal{M} \top$;
- (b) Not $\overline{\models}_\alpha^\mathcal{M} \top$.

COMMENT. $\top$ is a constant that is true at every world.

2

- (a) Not $\models_\alpha^\mathcal{M} \natural$;
- (b) Not $\overline{\models}_\alpha^\mathcal{M} \natural$.

COMMENT. $\natural$ is a constant that is neither true nor false at every world.

3

- (a) $\models_\alpha^\mathcal{M} (\varphi \leftrightarrow \psi)$ iff at least one of the following conditions are satisfied:
  - (i) $\models_\alpha^\mathcal{M} \varphi$ and $\models_\alpha^\mathcal{M} \psi$;
  - (ii) Not $\models_\alpha^\mathcal{M} \varphi$, not $\overline{\models}_\alpha^\mathcal{M} \varphi$, not $\models_\alpha^\mathcal{M} \psi$, and not $\overline{\models}_\alpha^\mathcal{M} \psi$;
  - (iii) $\overline{\models}_\alpha^\mathcal{M} \varphi$ and $\overline{\models}_\alpha^\mathcal{M} \psi$.
- (b) $\overline{\models}_\alpha^\mathcal{M} (\varphi \leftrightarrow \psi)$ iff one of the following conditions are satisfied:
  - (i) $\models_\alpha^\mathcal{M} \varphi$ and not $\models_\alpha^\mathcal{M} \psi$;
  - (ii) $\models_\alpha^\mathcal{M} \psi$ and not $\models_\alpha^\mathcal{M} \varphi$
  - (iii) not $\models_\alpha^\mathcal{M} \varphi$, not $\overline{\models}_\alpha^\mathcal{M} \varphi$, and $\models_\alpha^\mathcal{M} \psi$;
  - (iv) not $\models_\alpha^\mathcal{M} \varphi$, not $\overline{\models}_\alpha^\mathcal{M} \varphi$, and $\overline{\models}_\alpha^\mathcal{M} \psi$;
  - (v) $\overline{\models}_\alpha^\mathcal{M} \varphi$ and not $\overline{\models}_\alpha^\mathcal{M} \psi$;
  - (vi) not $\overline{\models}_\alpha^\mathcal{M} \varphi$ and $\overline{\models}_\alpha^\mathcal{M} \psi$.

COMMENT. $(\varphi \leftrightarrow \psi)$ is two-valued. In a given model $\mathcal{M}$, $(\varphi \leftrightarrow \psi)$ is true at the world α iff φ and ψ are either both true or both false at α or both neither true nor false at α. It is false at α otherwise. In other terms, $(\varphi \leftrightarrow \psi)$ is true if φ and ψ have the same truth conditions in $\mathcal{M}$ and it is false otherwise.

4

- (a) $\models_\alpha^\mathcal{M} \downarrow\varphi$ iff not $\models_\alpha^\mathcal{M} \varphi$ and not $\overline{\models}_\alpha^\mathcal{M} \varphi$;
- (b) $\overline{\models}_\alpha^\mathcal{M} \downarrow\varphi$ iff $\models_\alpha^\mathcal{M} \varphi$ or $\overline{\models}_\alpha^\mathcal{M} \varphi$.

COMMENT. $\downarrow \varphi$ is true at given world α iff φ is neither true nor false at α and it is false iff φ is either true or false at α. So, $\downarrow \varphi$ is a two-valued sentence. The semantics of '$\downarrow$' carries no modal import, being truth-functional.

5
(a) $\models_\alpha^{\mathcal{M}} \updownarrow \varphi$ iff $\models_\alpha^{\mathcal{M}} \varphi$ or $\overline{\models}_\alpha^{\mathcal{M}} \varphi$;
(b) $\overline{\models}_\alpha^{\mathcal{M}} \updownarrow \varphi$ iff not $\models_\alpha^{\mathcal{M}} \varphi$ and not $\overline{\models}_\alpha^{\mathcal{M}} \varphi$.

COMMENT. In a given model $\mathcal{M}$, $\updownarrow \varphi$ is true at given world α iff φ is either true or false at α and it is false at α iff φ is neither true nor false at α. So, $\updownarrow \varphi$ is a two-valued sentence. The semantics of '$\updownarrow$' carries no modal import, being truth-functional.

6
(a) $\models_\alpha^{\mathcal{M}} \Box\varphi$ iff for no β in $\mathcal{M}$ it holds that $\overline{\models}_\beta^{\mathcal{M}} \varphi$ and there exists β in $\mathcal{M}$ such that $\models_\beta^{\mathcal{M}} \varphi$;
(b) $\overline{\models}_\alpha^{\mathcal{M}} \Box\varphi$ iff there is β in $\mathcal{M}$ such that $\overline{\models}_\beta^{\mathcal{M}} \varphi$.

COMMENT. In a given model $\mathcal{M}$, $\Box\varphi$ is true iff φ is false at no world and there is word in $\mathcal{M}$ at which φ is true. $\Box\varphi$ is false if φ is false at some world. $\Box\varphi$ is neither true nor false only if φ is neither true nor false at every world. Note that $\Box\varphi$ does not *categorically* says that φ is necessarily true, but it *conditionally* says that φ is necessarily true under the condition that it is not the case that φ is neither true nor false at every world. Indeed, if φ is singular (see 2.6 below), neither condition (a) nor condition (b) is satisfied at any world, so that $\Box\varphi$ is neither true nor false at every world. However, one may express that φ is necessarily true by the formula $\Box{\uparrow}\varphi$.

Definition 2.5 (Singular sets of sentences in a model $\mathcal{M}$)

A finite set of sentences Γ is said to be singular in $\mathcal{M}$ iff for every world α in $\mathcal{M}$ and every sentence φ of Γ it holds that not $\models_\alpha^{\mathcal{M}} \varphi$ and not $\overline{\models}_\alpha^{\mathcal{M}} \varphi$.

Definition 2.6 (Singular sentences in a model $\mathcal{M}$)

A sentence φ is said to be singular in a model $\mathcal{M}$ iff the set $\{\varphi\}$ is singular.

Definition 2.7 (Satisfiability in a model $\mathcal{M}$)

A set Γ of sentences is satisfiable in the model $\mathcal{M}$ iff either (i) Γ is empty or (ii) Γ is singular in $\mathcal{M}$, or (iii) for every nonempty subset Γ' of Γ there is a world α in $\mathcal{M}$ such that for every element φ in Γ' it holds that not $\overline{\models}_\alpha^{\mathcal{M}} \varphi$ and there is an element ψ in Γ' such that $\models_\alpha^{\mathcal{M}} \psi$.

COMMENT. Singular sets are included among satisfiable sets by pure convention. Unless using a partial logic even at the metalinguistic level, we have to decide whether singular sets are satisfiable or unsatisfiable. Our choice presents some technical advantage, but it is otherwise harmless.

Definition 2.8 (Logical satisfiability)

A set Γ of sentences is logically satisfiable iff it is satisfiable in every model $\mathcal{M}$.

Definition 2.9 (Unsatisfiability in a model $\mathcal{M}$)

A set Γ of sentences is unsatisfiable in the model $\mathcal{M}$ iff Γ is not satisfiable in $\mathcal{M}$.

Definition 2.10 (Logical unsatisfiability)

A set Γ of sentences is logically unsatisfiable iff it is unsatisfiable in every model.

Theorem 2.2 (Compactness fails)

There are a model $\mathcal{M}$ and an infinite set A of sentences such that every finite subset of A is satisfiable while A is not satisfiable.

Proof
Assume that $\mathcal{M}$ contains a denumerable ordered set B of atomic sentences $\mathbb{P}_{i_0}$, $\mathbb{P}_{i_1}$,... such that for every i_j every world belongs to P_{i_j} (so that every element of B is two-valued in $\mathcal{M}$) and for at least a k, $Q_{i_k} \neq \emptyset$. Now, let C be the set of all sentences of the form $(\neg\mathbb{P}_{i_j} \wedge \mathbb{P}_{i_{j+1}}) \mid (\mathbb{P}_{i_j} \vee \mathbb{P}_{i_{j+1}}))$. We prove:

(a) C is not logically satisfiable;

(b) every finite subset of C is satisfiable.

(a) By hypothesis the set of worlds in $\mathcal{M}$ at which at least one element of B is true is not empty. Consider the set V of such worlds. At every world, if $\mathbb{P}_{i_j}$ is true then $(\neg\mathbb{P}_{i_j} \wedge \mathbb{P}_{i_{j+1}}) \mid (\mathbb{P}_{i_j} \vee \mathbb{P}_{i_{j+1}}))$ is false by definition 2.4. Now, at every world $\in V$, there is a k such that $(\neg\mathbb{P}_{i_k} \wedge \mathbb{P}_{i_{k+1}}) \mid (\mathbb{P}_{i_k} \vee \mathbb{P}_{i_{k+1}}))$ is false. This proves that C, by definition 2.10 is not logically satisfiable.

(b) Let A be a finite subset $\mathbb{P}_{i_1}, \ldots, \mathbb{P}_{i_r}$ of B ($r \geq 2$). Let $\mathcal{M}$ be such that at some world α in $\mathcal{M} \models^{\mathcal{M}}_{\alpha} \mathbb{P}_{i_r}$ and for every w such that $1 \leq w < r$ it holds that $\overline{\models}^{\mathcal{M}}_{\alpha}\ \mathbb{P}_{i_w}$. By definition 2.4 for every w such that $1 \leq w < r-1$ $(\neg\mathbb{P}_{i_w} \wedge \mathbb{P}_{i_{w+1}}) \mid (\mathbb{P}_{i_w} \vee \mathbb{P}_{i_{w+1}}))$ is neither true nor false at α, while $\models^{\mathcal{M}}_{\alpha}$ $(\neg\mathbb{P}_{i_{r-1}} \wedge \mathbb{P}_{i_r}) \mid (\mathbb{P}_{i_{r-1}} \vee \mathbb{P}_{i_r}))$. Therefore, by definition 2.8 is logically satisfiable.

q.e.d.

COMMENT. The example used in proving theorem 2.2 is adapted from Adams 1975: 51-2. In the light of theorem 2.2 no axiomatic system would be complete if the notion of logical consequence is also defined for infinite sets of premises. In any case, since in this paper, we aim at providing a truth-conditional counterpart of Adams' probabilistic semantics, which is defined only with respect to finite sets of sentences, in what follows we'll consider only finite sets of sentences as Adams does.

Definition 2.11 (Validity in a model $\mathcal{M}$)

φ is valid in the model $\mathcal{M}$ ($\models_{\mathcal{M}}$) iff for every world α in $\mathcal{M}$, $\models^{\mathcal{M}}_{\alpha} \varphi$.

Definition 2.12 (Countervalidity in a model $\mathcal{M}$)

φ is countervalid in the model $\mathcal{M}$ ($\overline{\models}_{\mathcal{M}}$) iff for every world α in $\mathcal{M}$, $\overline{\models}_{\alpha}^{\mathcal{M}}$ φ

Definition 2.13 (Logical validity)

φ is valid ($\models \varphi$) iff for every model $\mathcal{M}$ and every world α in $\mathcal{M}$, $\models_{\alpha}^{\mathcal{M}} \varphi$.

COMMENT. A valid sentence is an instance of a valid schema. In other terms, every sentence obtained by replacing some sub-sentences of a valid sentence with other sentences is again a valid sentence. Hence the qualification 'logical'.

Definition 2.14 (Logical countervalidity)

φ is logically countervalid. For every model $\mathcal{M}$ and every world α in $\mathcal{M}$, $\overline{\models}_{\alpha}^{\mathcal{M}}$ φ.

COMMENT. A logically countervalid sentence is an instance of a countervalid schema. In other terms, every sentence obtained by replacing some sub-sentences of a countervalid sentence with other sentences is again a countervalid sentence.

Definition 2.15 ($\mathcal{M}$–consequence)

φ is an $\mathcal{M}$-consequence of the finite set Γ of sentences ($\Gamma \models_{\mathcal{M}} \varphi$) iff either $\Gamma = \emptyset$ and $\models_{\mathcal{M}} \varphi$ or Γ is unsatisfiable in $\mathcal{M}$ or there exists a subset $\Gamma' = \{\varphi_1, \ldots, \varphi_k\} (k > 0)$ of Γ such that for every world α in $\mathcal{M}$ it holds that: (i) if every $\psi \in \Gamma'$ is not false at α then φ is not false at α, (ii) if every $\psi \in \Gamma'$ is not false at α and at least an element of Γ' is true at α then φ is true at α.

Definition 2.16 ($\mathcal{M}$–equivalence)

The sentences φ and ψ are $\mathcal{M}$–equivalent iff $\{\varphi\} \models_{\mathcal{M}} \psi$ and $\{\psi\} \models_{\mathcal{M}} \varphi$.

Definition 2.17 ($\mathcal{M}$–contingency)

The sentence φ is $\mathcal{M}$–contingent iff there are two worlds α and β in $\mathcal{M}$ such that $\models_{\alpha}^{\mathcal{M}} \varphi$ and $\overline{\models}_{\beta}^{\mathcal{M}}$ φ

Definition 2.18 ($\mathcal{M}$–compatibility)

The sentences φ and ψ are $\mathcal{M}$–compatible iff either both φ and ψ are singular or there is a world α in $\mathcal{M}$ at which $\models_{\alpha}^{\mathcal{M}} \varphi$ and $\models_{\alpha}^{\mathcal{M}} \psi$.

Definition 2.19 (Logical consequence)

φ is a logical consequence of the finite set Γ of sentences ($\Gamma \models \varphi$) iff for every model $\mathcal{M}$, $\Gamma \models_{\mathcal{M}} \varphi$.

Definition 2.20 (Logical equivalence)

The sentences φ and ψ are logically equivalent iff $\{\varphi\} \models \psi$ and $\{\psi\} \models \varphi$.

Theorem 2.3

Two sentences φ and ψ are logically equivalent iff for every model $\mathcal{M}$ and every world α in $\mathcal{M}$ it holds that $\models_{\alpha}^{\mathcal{M}} (\varphi \leftrightarrow \psi)$.

Proof
Omitted.

q.e.d.

COMMENT. Two logically equivalent sentences have the same truth conditions in every model.

3. Some Fundamental Theorems

Theorem 3.1

If Γ is a finite set of sentences and φ is a nonsingular sentence, then $\Gamma \models_{\mathcal{M}} \varphi$ iff $\Gamma \cup \{\neg\varphi\}$ is unsatisfiable in $\mathcal{M}$.

Proof
Suppose $\Gamma \models_{\mathcal{M}} \varphi$. If Γ is empty then $\models_{\mathcal{M}} \varphi$, so that for every world α it holds that $\overline{\models}_{\alpha}^{\mathcal{M}} \neg\varphi$. In this case, $\{\varphi\}$ is not satisfiable in $\mathcal{M}$. If Γ is unsatisfiable in $\mathcal{M}$, also $\Gamma \cup \{\neg\varphi\}$ is unsatisfiable in $\mathcal{M}$. If Γ is not empty and it is satisfiable in $\mathcal{M}$, suppose that $\Gamma \cup \{\neg\varphi\}$ is satisfiable in $\mathcal{M}$. For every nonempty subset Γ' of $\Gamma \cup \{\neg\varphi\}$ there is a world α in $\mathcal{M}$ such that for every element ψ in Γ' it holds that not $\overline{\models}_{\alpha}^{\mathcal{M}} \psi$. Moreover, there is an element χ in Γ' such that $\models_{\alpha}^{\mathcal{M}} \chi$. Since, by hypothesis, $\Gamma \models_{\mathcal{M}} \varphi$, there is a subset Δ of Γ and a subset S of worlds in $\mathcal{M}$ such that (i) for at least a sentence $\gamma \in \Delta$ and every world $\alpha \in S$ it holds that $\models_{\alpha}^{\mathcal{M}} \gamma$ and (ii) for no sentence $\gamma \in \Delta$ it holds that $\overline{\models}_{\alpha}^{\mathcal{M}} \gamma$. Therefore, for every world $\alpha \in S$, it holds that $\models_{\alpha}^{\mathcal{M}} \varphi$, so that $\overline{\models}_{\alpha}^{\mathcal{M}} \neg\varphi$ for each $\alpha \in S$. Moreover, at every world $\alpha \notin S$ there is a sentence $\gamma \in \Delta$ such that $\overline{\models}_{\alpha}^{\mathcal{M}} \gamma$. This contradicts that $\Gamma \cup \{\neg\varphi\}$ is satisfiable in $\mathcal{M}$.

Suppose now that $\Gamma \cup \{\neg\varphi\}$ is unsatisfiable in $\mathcal{M}$. If Γ is unsatisfiable in $\mathcal{M}$ then $\Gamma \models_{\mathcal{M}} \varphi$. So, suppose that Γ is satisfiable. If Γ is empty, then $\{\neg\varphi\}$ is unsatisfiable in $\mathcal{M}$, so that φ is valid in $\mathcal{M}$. Again, it holds that $\Gamma \models_{\mathcal{M}} \varphi$. If Γ is singular and φ is singular too, it holds again that $\Gamma \models_{\mathcal{M}} \varphi$. If Γ is singular and φ is not singular, then $\{\neg\varphi\}$ is unsatisfiable in $\mathcal{M}$, so that $\models_{\mathcal{M}} \varphi$. Again it holds that $\Gamma \models_{\mathcal{M}} \varphi$. Suppose now that Γ is nonempty and nonsingular. By definition, being Γ satisfiable in $\mathcal{M}$, for every nonempty subset Γ' of Γ there is at least one world α in $\mathcal{M}$ such that for every element ψ in Γ' it holds that not $\overline{\models}_{\alpha}^{\mathcal{M}} \psi$ and there is an element χ in Γ' such that $\models_{\alpha}^{\mathcal{M}} \chi$. If $\Gamma \cup \{\neg\varphi\}$ is unsatisfiable in $\mathcal{M}$, for every such world α must be $\overline{\models}_{\alpha}^{\mathcal{M}} \neg\varphi$, so that it holds that $\models_{\alpha}^{\mathcal{M}} \varphi$ and also $\Gamma \models_{\mathcal{M}} \varphi$.

q.e.d.

Theorem 3.2

Let $\mathcal{M} = (W, P, Q)$ be a model and let φ be a sentence.

(A) If φ is true at some world α, and is false at no world, so that for every β in $\mathcal{M}$ it holds that not $\overline{\models}_{\beta}^{\mathcal{M}}$ φ, then φ is valid in $\mathcal{M}$;

(B) If φ is false at some world α, so that $\overline{\models}_{\alpha}^{\mathcal{M}}$ φ and is true at no world, so that for every β in $\mathcal{M}$ it holds that not $\models_{\beta}^{\mathcal{M}}$ φ, then φ is false at every world in $\mathcal{M}$.

Proof

Let us proceed by induction on the construction of φ.

1. φ is an atomic sentence $\mathbb{P}_n$.

   (A) Since there is a world at which φ is true, it holds that $Q_n \neq \emptyset$. Since φ is false at no world it holds that $P_n = Q_n$. Suppose that φ is not true at any world β. Then $\beta \notin Q_n$. But since $P_n = Q_n$ and $Q_n \neq \emptyset$, by definition 2.4, φ is true at β. So, by *consequentia mirabilis* φ is true at β. So, it is true at all worlds in $\mathcal{M}$. We conclude: $\models_{\mathcal{M}} \varphi$.

   (B) Since there is a world at which φ is false, it holds that $P_n \neq \emptyset$. Since φ is true at no world it holds that $Q_n = \emptyset$. Suppose φ is not false at any world β. So, $\beta \notin P_n$. But since $Q_n = \emptyset$, by definition 2.4, φ is false at β. So, by *consequentia mirabilis* φ is false at β. So, it is false at all worlds. We conclude: $\overline{\models}_{\mathcal{M}} \varphi$.

2. φ = '$\bot$'.

   (A) In this case $\overline{\models}_{\mathcal{M}} \varphi$, so that the condition in the theorem is vacuously satisfied.

   (B) Trivial.

3. $\varphi \approx \uparrow\psi$.

   (A) Since $\uparrow\psi$ is a two-valued sentence, if φ is false at no world, then it is true at every world, so that $\models_{\mathcal{M}} \varphi$.

   (B) Since $\uparrow\psi$ is a two-valued sentence, if φ is true at no world, then it is false at every world, so that $\overline{\models}_{\mathcal{M}} \varphi$.

4. $\varphi \approx \neg\psi$.

   (A) Suppose that φ is true at some world in $\mathcal{M}$, and that φ is false at no world in $\mathcal{M}$. In this case, ψ is false at some world and true at no world in $\mathcal{M}$. By inductive hypothesis ψ is false at all worlds in $\mathcal{M}$. Then φ is true at all worlds in $\mathcal{M}$ and hence $\models_{\mathcal{M}} \varphi$.

   (B) Suppose φ is false at some world in $\mathcal{M}$, and that φ is true at no world in $\mathcal{M}$. In this case, ψ is true at some world in $\mathcal{M}$ and false at no world in $\mathcal{M}$. By inductive hypothesis $\models_{\mathcal{M}} \psi$. Then φ is false at all worlds in $\mathcal{M}$ and hence $\overline{\models}_{\mathcal{M}} \varphi$.

5. $\varphi \approx (\psi \vee \chi)$.

   (A) Suppose φ is true at some world α in $\mathcal{M}$, and that φ is false at no world in $\mathcal{M}$. Suppose that φ is not true at a given world β in $\mathcal{M}$. Neither ψ nor χ are therefore true at β. Now, ψ and χ are not both false at β, otherwise φ would be false at β. So at least one of them (say ψ) is neither true nor false at β. However, either ψ is true at α or χ is true at α or by 2.4 there is a world γ in $\mathcal{M}$ at which either ψ or χ are true. In this last case, by 2.4 ψ and χ are not both false at γ. From this follows by 2.4 that φ is true at β. Contradiction.

   (B) Suppose φ is false at some world α in $\mathcal{M}$, and that φ is true at no world in $\mathcal{M}$. Suppose that φ is not false at a given world β in $\mathcal{M}$. Either ψ or χ are therefore not false at β. Now, either ψ or χ are not true at β, otherwise φ would be true at β. So at least one of them (say ψ) is neither true nor false at β. However, either both ψ and χ are false at α or, in any case, by 2.4 there is a world γ in $\mathcal{M}$ at which both ψ and χ are false. Moreover, there is no world γ in $\mathcal{M}$ at which either ψ or χ are true because in such a case φ would be true at γ. From this follows by 2.4 that φ is false at β. Contradiction.

6. $\varphi \approx (\psi \wedge \chi)$.

   (A) Suppose φ is true at some world α in $\mathcal{M}$, and that φ is false at no world in $\mathcal{M}$. Suppose that φ is not true at a given world β in $\mathcal{M}$. Either ψ or χ (or both) are not true at β. It cannot be $\overline{\models}_{\beta}^{\mathcal{M}} \psi$ or $\overline{\models}_{\beta}^{\mathcal{M}} \chi$. So, it holds not $\overline{\models}_{\beta}^{\mathcal{M}} \psi$ and not $\overline{\models}_{\beta}^{\mathcal{M}} \chi$. Then, by definition 2.4 for every world γ in $\mathcal{M}$ it holds that $\models_{\gamma}^{\mathcal{M}} \psi$ and $\models_{\gamma}^{\mathcal{M}} \chi$, so that $\models_{\mathcal{M}} \varphi$.

   (B) Suppose that φ is false at some world α in $\mathcal{M}$, and that φ is true at no world in $\mathcal{M}$. So at some world α either $\overline{\models}_{\alpha}^{\mathcal{M}} \varphi$ or $\overline{\models}_{\alpha}^{\mathcal{M}} \psi$. Moreover, for every world β it holds that not $\models_{\beta}^{\mathcal{M}} \varphi$ and not $\models_{\beta}^{\mathcal{M}} \psi$. Therefore, by definition 2.4, it holds that $\overline{\models}_{\beta}^{\mathcal{M}} \varphi$. So φ is false at every world γ in $\mathcal{M}$ and hence it holds that $\overline{\models}_{\mathcal{M}} \varphi$.

7. $\varphi \approx (\psi \mid \chi)$.

   (A) Suppose φ is true at some world α in $\mathcal{M}$, and that φ is false at no world in $\mathcal{M}$. Suppose that φ is not true at a given world β in $\mathcal{M}$. Either ψ or χ (or both) are not true at β. Now, under our assumption is not possible that, at β, χ is true and ψ is false, otherwise φ would be false at β. So, either χ is not true at β or ψ is not false at β. However, either both ψ and χ are true at α or by definition 2.4, there is a world γ in $\mathcal{M}$ such that both χ and ψ are true at γ. By definition 2.4 it is not the case that χ is true at γ and ψ false at γ. From this it follows, by definition 2.4, that φ is true at β. Contradiction.

(B) Suppose φ is false at some world α in $\mathcal{M}$, and that φ is true at no world in $\mathcal{M}$. Suppose that φ is not false at a given world β in $\mathcal{M}$. Therefore, it is not the case that χ is true and ψ false at β. Moreover, it is not the case that both ψ and χ are true at β, otherwise φ would be true at β. However, either χ is true at α and ψ is false at α or, by condition definition 2.4 there is a world γ in $\mathcal{M}$, at which χ is true and ψ is false. Moreover, there is no world γ in $\mathcal{M}$ at which both χ and ψ are true because in such a case φ would be true at γ. From this follows by definition 2.4 that φ is false at β. Contradiction.

8. $\varphi \approx (\psi \rightarrow \chi)$.

(A) Suppose φ is true at some world α in $\mathcal{M}$, and that φ is false at no world in $\mathcal{M}$. Since, at each world in $\mathcal{M}$, φ is either true or false, φ is true at all worlds in $\mathcal{M}$, so that it holds that $\models_{\mathcal{M}} \varphi$.

(B) Suppose φ is false at some world α in $\mathcal{M}$, and that φ is true at no world in $\mathcal{M}$. Since, at each world in $\mathcal{M}$, φ is either true or false, φ is false at all worlds in $\mathcal{M}$, so that it holds that $\overline{\models}_{\mathcal{M}} \varphi$.

9. $\varphi \approx \Diamond\psi$.

(A) Suppose φ is true at some world α in $\mathcal{M}$, and that φ is false at no world in $\mathcal{M}$. Then, by definition 2.4, φ is true at every world α in $\mathcal{M}$, so that it holds that $\models_{\mathcal{M}} \varphi$.

(B) Suppose φ is false at some world α in $\mathcal{M}$, and that φ is true at no world in $\mathcal{M}$. Then, by definition 3, φ is false at every world α in $\mathcal{M}$, so that it holds that $\overline{\models}_{\mathcal{M}} \varphi$.

q.e.d.

COMMENT. In the light of theorem 3.2 quasi-tautologies (that is sentences that at each world α in each model $\mathcal{M}$ are either neither true nor false or true) are singular or valid. They are near-tautologies (see definition 4.2 below.)

Theorem 3.3

Let Γ be a finite set of sentences. Let φ be a sentence. Then $\Gamma \models_{\mathcal{M}} \varphi$ iff $\Gamma \cup \top \models_{\mathcal{M}} \varphi$.

Proof

Suppose that $\Gamma \cup \top \models_{\mathcal{M}} \varphi$. If Γ is empty then $\top \models_{\mathcal{M}} \varphi$, so that φ is $\mathcal{M}$–valid. In this case $\Gamma \models_{\mathcal{M}} \varphi$. Otherwise, let Γ' be a non-empty subset of $\Gamma \cup \top$ such that for every world α in $\mathcal{M}$ it holds that: (i) if every $\psi \in \Gamma'$ is not false at α then φ is not false at α, (ii) if every $\psi \in \Gamma'$ is not false at α and at least an element of Γ' is true at α then φ is true at α. There are two cases: (i) $\top \in \Gamma'$, and (ii) $\top \notin \Gamma'$. In the case (i) φ is $\mathcal{M}$–valid, so that $\Gamma \models_{\mathcal{M}} \varphi$. In the case (ii) it is immediate that $\Gamma \models_{\mathcal{M}} \varphi$.

Suppose $\Gamma \models_{\mathcal{M}} \varphi$. If $\Gamma = \emptyset$ then φ is $\mathcal{M}$–valid, so that $\top \models_{\mathcal{M}} \varphi$ and $\Gamma \cup \top \models_{\mathcal{M}} \varphi$. If $\Gamma \neq \emptyset$, then there is a non-empty subset Γ' of Γ such that (i) if every $\psi \in \Gamma'$ is not false at α then φ is not false at α, (ii) if every $\psi \in \Gamma'$ is not false at α and at least one element of Γ' is true at α, then φ is true at α. Now, Γ' is also a subset of $\Gamma \cup \top$, so that $\Gamma \cup \top \models_{\mathcal{M}} \varphi$.

q.e.d.

4. Quasi S5 Modal System

4.1 Classic Tautologies and Near Tautologies

Definition 4.1 (Classic tautologies)

Let $\mathscr{S}$ be the set of sentences of $\mathscr{L}$ in which the logical constants that occur in them are in the set {'$\bot$', '$\top$', '$\neg$', '$\vee$', '$\wedge$'}. Let $\mathscr{T}$ be the set of sentential tautologies belonging to $\mathscr{S}$. Any element of $\mathscr{S}$ is called a *classic tautology*.

Classic tautological schemas are not, in general, valid schemas in our theory. It is a well-known result that in three-valued logic, tautologies are quasi-tautologies. This result means that instances of tautologous schemas, at every world in every model, are not false, but they may be either true or null at any world. In our theory, we may further sharp this result: every element of $\mathscr{T}$ is either logically valid or logically singular (that is neither true nor false at every world in every model). We call those sentences that have this property *near-tautologies*. Keep in mind that the property of being a near-tautology refers not to a single model but to the set of all models (so that they are instances of near-tautologous schemas).

Definition 4.2 (Near-tautologies)

A sentence φ of $\mathscr{L}$ is called a near-tautology iff for every model $\mathcal{M}$, either $\models_{\mathcal{M}} \varphi$ or φ is singular in $\mathcal{M}$.

Theorem 4.1

Let φ be a classic tautology. Then φ is a near-tautology.

Proof

If φ is a classic tautology then, for every model $\mathcal{M}$ and every world in $\mathcal{M}$ it holds that not $\overline{\models}_{\alpha}^{\mathcal{M}} \varphi$. Now, there are two cases: (a) there is at least a world α in $\mathcal{M}$ at which $\models_{\alpha}^{\mathcal{M}} \varphi$ and (b) for each world α in $\mathcal{M}$ it holds that not $\models_{\alpha}^{\mathcal{M}} \varphi$. In the case (a) by theorem 3.2 $\models_{\mathcal{M}} \varphi$; in the case (b), since it holds that not $\models_{\alpha}^{\mathcal{M}} \varphi$ and not $\overline{\models}_{\alpha}^{\mathcal{M}} \varphi$, φ is a singular sentence. Hence φ is a near-tautology.

q.e.d.

Example. Consider a sentence of the form $(\varphi \vee \neg\varphi)$. If there is a world α in a model $\mathcal{M}$ such that $\models_{\alpha}^{\mathcal{M}} \varphi$ or $\overline{\models}_{\alpha}^{\mathcal{M}} \varphi$ it holds that $\models_{\mathcal{M}} \varphi$. If such a world and model do not exist, then φ is a singular sentence. This is the case when φ is logically equivalent to $\natural$ or $\varphi \approx \chi \mid \psi$, where ψ is logically equivalent to $\bot$.

Near-tautologies do not instance valid schemas. However, they are valid whenever they are not singular sentences. Since a singular sentence logically entails a valid sentence and '$\natural$' is a singular sentence, we have the following result:

Theorem 4.2

Let φ be a near-tautology. Then $\natural \models \varphi$.

Proof

There are two cases. (a) φ is logically valid, and (b) φ is logically singular. In the case (a) for every sentence ψ it holds that $\psi \models \varphi$. In particular, it holds that $\natural \models \varphi$. In the case (b) '$\natural$' and φ have the same truth conditions, so that, $\natural \models \varphi$.

q.e.d.

4.2 Weak Deduction Theorem

Deduction theorem cannot be, in general, satisfied in the present theory. This result is due mainly to the fact that the algebraic structure that underlies our semantics, not being a distributive lattice, is *not* a Heyting algebra. In this case, no material implication satisfying *modus ponens* exists. Despite this, our material implication (that returns a two-valued sentence) satisfies the following weaker property:

Theorem 4.3

For every pair of sentences φ, ψ, and every model $\mathcal{M}$ it holds that $\varphi \models_{\mathcal{M}} \psi$ iff $\models_{\mathcal{M}} (\varphi \rightarrow \psi)$.

Proof

Suppose that $\varphi \models_{\mathcal{M}} \psi$. By definition 2.15, at every world α in $\mathcal{M}$, either $\models_{\alpha}^{\mathcal{M}} \varphi$ and $\models_{\alpha}^{\mathcal{M}} \psi$ or not $\overline{\models}_{\alpha}^{\mathcal{M}} \varphi$ and not $\overline{\models}_{\alpha}^{\mathcal{M}} \psi$ or $\overline{\models}_{\alpha}^{\mathcal{M}} \varphi$ and $\overline{\models}_{\alpha}^{\mathcal{M}} \psi$. In each of these case, it holds, by definition 2.4 that $\models_{\alpha}^{\mathcal{M}} (\varphi \rightarrow \psi)$, so that $\models_{\mathcal{M}} (\varphi \rightarrow \psi)$.

Suppose that $\models_{\mathcal{M}} (\varphi \rightarrow \psi)$. Then at every world α in $\mathcal{M}$ it holds that $\models_{\alpha}^{\mathcal{M}} (\varphi \rightarrow \psi)$. By definition 2.4, either $\models_{\alpha}^{\mathcal{M}} \varphi$ and $\models_{\alpha}^{\mathcal{M}} \psi$ or not $\overline{\models}_{\alpha}^{\mathcal{M}} \varphi$ and not $\overline{\models}_{\alpha}^{\mathcal{M}} \psi$ or $\overline{\models}_{\alpha}^{\mathcal{M}} \varphi$ and $\overline{\models}_{\alpha}^{\mathcal{M}} \psi$. By definition 2.15, it holds that $\varphi \models_{\mathcal{M}} \psi$.

q.e.d.

It should be noted that, in general, *modus ponens* does not hold concerning material conditional $\rightarrow$. However, *modus ponens* is always valid if either the antecedent or the consequent is a two-valued sentence (see *infra*). In particular, if φ is valid, from φ and $(\varphi \rightarrow \psi)$ follows ψ. So, a deductive system for valid formulas, based on *modus ponens*, is in theory possible.

4.3 Two-Valued Sentences

Along with tri-events that may be at the same time neither true nor false, there are *two-valued* sentences, expressed by sentences that comply with the excluded-middle principle. A sentence may be two-valued in one model $\mathcal{M}$ without being two-valued in all models. Even an atomic sentence $\mathbb{P}_n$ may be two-valued in a model $\mathcal{M} = (W, P, Q)$ if $P_n = W$. A molecular sentence whose atomic sentences are two-valued and whose logical constants occurring in it belong to the set {'$\bot$', '$\vee$', '$\neg$', '$\wedge$','$\rightarrow$', '$\Diamond$', '$\uparrow$'} (i.e. the set of all the primitive constants minus {|} is, in turn, two-valued. A lattice of "genuine" tri-events may be generated using conditioning '|' along with the other logical constants from a set of propositions expressed by two-valued sentences. The notion of a two-valued sentence is made precise by the following definition:

Definition 4.3 (Two-valued sentences)

A sentence φ of $\mathscr{L}$ is called a two-valued sentence in a model $\mathcal{M}$ iff for every world α in $\mathcal{M}$ either $\models_{\alpha}^{\mathcal{M}} \varphi$ or $\overline{\models}_{\alpha}^{\mathcal{M}} \varphi$

Given a model $\mathcal{M}$, the set of the two-valued sentences of $\mathscr{L}$ in $\mathcal{M}$ is closed concerning the connectives $\neg$, $\vee$, and $\wedge$. Since concerning such sentences, these connectives are Boolean, the set of those propositions that express two-valued sentences of $\mathscr{L}$ in $\mathcal{M}$ form a Boolean algebra.

The notion of a sentence which is two-valued in a given model $\mathcal{M}$, cannot be characterised by syntactic means. Indeed, in a given model $\mathcal{M}$, this property may depend on the truth conditions of atomic sentences in $\mathcal{M}$.

4.4 Essentially Two-Valued Sentences

By the term 'essentially two-valued sentence' we mean a sentence that is two-valued in every model.

Definition 4.4 (Essentially two-valued sentences)

A sentence φ of $\mathscr{L}$ is called an essentially two-valued sentence iff it is a two-valued sentence in every model.

4.5 Ordinary Sentences

Is there any syntactical counterpart of the notion of 'essentially two-valued sentence'? The following results show that the answer is in the positive: there is a recursively syntactically defined class $\mathscr{O}$ of sentences (thereby called *ordinary sentences*), closed under logical constants except $\natural$ and $|$, whose elements are essentially two-valued. Moreover, for every essentially two-valued sentence, there is an ordinary sentence that is logically equivalent to it.

Definition 4.5 (Ordinary sentences)

φ is an ordinary sentence iff at least one among the following conditions is satisfied :

(a) $\varphi \approx \bot$

(b) $\varphi \approx \uparrow\psi$

(c) $\varphi \approx (\psi \rightarrow \chi))$;

(d) $\varphi \approx \Diamond\psi$ and ψ is an ordinary sentence;

(e) $\varphi \approx \neg\psi$ and ψ is an ordinary sentence;

(f) $\varphi \approx (\psi \vee \chi)$ and both ψ and χ are ordinary sentences;

(g) $\varphi \approx (\psi \wedge \chi)$ and both ψ and χ are ordinary sentences;

By derived logical constants, the following result holds.

Theorem 4.4

Let φ and ψ be ordinary sentences. Then the following are also ordinary sentences: '$\top$', $\downarrow\varphi$, $(\varphi \leftrightarrow \psi)$, $\Box\varphi$

Proof
Omitted.

q.e.d.

Since the class $\mathscr{O}$ of the ordinary sentences is closed under all the logical constants except $\natural$ and $|$, we may use special meta-variables $\hat{\varphi}, \hat{\psi}, \hat{\chi}$, and so on for expressing schemas that refer to the class of ordinary sentences. A valid schema in which occur such meta-variables is meant to be valid in the sense that every sentence obtained by substitution of such meta-variables with ordinary sentences is true at every world in every model. We will call such schemas O-valid schemas.

We should prove that the set of ordinary sentences, as defined by definition 4.5, semantically coincides with the set of essentially two-valued sentences. The following result proves this.

Theorem 4.5

A sentence φ is an essentially two-valued sentence iff it is logically equivalent to an ordinary sentence.

Proof
If φ is an essentially two-valued sentence, so that for every model $\mathcal{M}$ and every world α in $\mathcal{M}$ it holds that either φ or $\neg\varphi$, then the sentence $\uparrow\varphi$ is logically equivalent to φ and it is an ordinary sentence.

If φ is logically equivalent to an ordinary sentence ψ, it is straightforward, by definition 4.5, definition 2.4 and theorem 2.1 that at every world α in every model $\mathcal{M}$, it holds: either ψ or $\neg\psi$, so that ψ is an essentially two-valued sentence.

q.e.d.

5. Normal Forms

In this section, we will prove first that every sentence of $\mathscr{L}$ is logically equivalent to a syntactically simple sentence. More precisely, we show that every sentence is logically equivalent to a sentence of the form $(\psi \mid \varphi)$, where both ψ and φ are essentially two-valued sentences, possibly containing occurrences of modal symbols. Second, we will prove that to every sentence φ of $\mathscr{L}$ we may effectively associate an S5-valid sentence φ' of the standard modal language $\mathscr{L}_2$ (which is a sub-language of $\mathscr{L}$) so that φ is valid iff φ' is S5-valid. To prove these results we need to prove several general results and to introduce the idea of the "normal quasi-classical form" and to prove the every essentially two-valued sentence is logically equivalent to a sentence in such a normal form.

Theorem 5.1

Let φ be a sentence of $\mathscr{L}$. At every world α in every model $\mathcal{M}$ it holds: iff

1. $\models^{\mathcal{M}}_{\alpha} \uparrow\varphi$ iff $\overline{\models}^{\mathcal{M}}_{\alpha} \uparrow\neg\varphi$
2. $\overline{\models}^{\mathcal{M}}_{\alpha} \uparrow\varphi$ iff $\models^{\mathcal{M}}_{\alpha} \uparrow\neg\varphi$

Proof
Omitted.

q.e.d.

Theorem 5.2

Double negation law. Let φ be a sentence of $\mathscr{L}$. The following equivalence holds: $\models \varphi \leftrightarrow \neg\neg\varphi$

Proof
Let $\mathcal{M}$ be a model and α any world in $\mathcal{M}$. Suppose $\models^{\mathcal{M}}_{\alpha} \varphi$. Then, by definition 2.4 $\overline{\models}^{\mathcal{M}}_{\alpha} \neg\varphi$ and also $\models^{\mathcal{M}}_{\alpha} \neg\neg\varphi$. Suppose that $\overline{\models}^{\mathcal{M}}_{\alpha} \varphi$. Then, by definition 2.4 $\models^{\mathcal{M}}_{\alpha} \neg\varphi$ and also $\overline{\models}^{\mathcal{M}}_{\alpha} \neg\neg\varphi$. Suppose that not $\models^{\mathcal{M}}_{\alpha} \varphi$ and not $\overline{\models}^{\mathcal{M}}_{\alpha} \varphi$. Then, by definition 2.4, not $\models^{\mathcal{M}}_{\alpha} \neg\varphi$ and not $\overline{\models}^{\mathcal{M}}_{\alpha} \neg\varphi$ and also not $\models^{\mathcal{M}}_{\alpha} \neg\neg\varphi$ and not $\overline{\models}^{\mathcal{M}}_{\alpha} \neg\neg\varphi$. So φ and $\neg\varphi$ have the same truth-value at α, so that, by theorem 2.1 $\models^{\mathcal{M}}_{\alpha} \varphi \leftrightarrow \neg\neg\varphi$. Since this holds for every world in every model, it holds that $\models \varphi \leftrightarrow \neg\neg\varphi$.

q.e.d.

5.1 Simple Tri-events

A simple conditional is a sentence of the form 'if A then C' where A and C are categorical sentences. Our language $\mathscr{L}$ contains two conditional constants: the particular two-valued material implication ('$\rightarrow$') and conditioning ('$|$'). Only the latter is assumed to represent those conditionals that pass the Ramsey-test. So, we characterise simple tri-events as those sentences of $\mathscr{L}$ having the form $\psi \mid \varphi$, where both ψ and φ do not contain any occurrence of the conditioning symbol '$|$' (nor any of the defined connectives defined in terms of '$|$'). This characterisation makes sense only if '$|$' cannot be defined in terms of the other primitive connectives. This is proved by the following result.

Theorem 5.3

There are a model $\mathcal{M}$ and two sentences φ and ψ of $\mathscr{L}$ containing only primitive symbols such that no sentence χ of $\mathscr{L}$ that does not contain any occurrence of the symbol '$|$' is $\mathcal{M}$-equivalent to $(\psi \mid \varphi)$.

Proof

Let φ and ψ be two sentences of $\mathscr{L}$ (containing only primitive symbols) and let $\mathcal{M}$ be a model. Let us assume, without loss of generality that:

(a) φ, ψ, and $\mathcal{M}$ are such that both φ and ψ are two valued in $\mathcal{M}$, ψ is not a $\mathcal{M}$–consequence of $\{\varphi\}$, φ is not a $\mathcal{M}$–consequence of $\{\psi\}$, both φ and ψ are $\mathcal{M}$–contingent sentences, both φ and ψ are $\mathcal{M}$–compatible sentences, so that

  (i) at each world α in $\mathcal{M}$ either $\models^{\mathcal{M}}_{\alpha} \varphi$ or $\overline{\models}^{\mathcal{M}}_{\alpha} \varphi$ and either $\models^{\mathcal{M}}_{\alpha} \psi$ or $\overline{\models}^{\mathcal{M}}_{\alpha} \psi$

  (ii) there are worlds α, β, γ, δ all in $\mathcal{M}$, such that $\models^{\mathcal{M}}_{\alpha} \varphi$, $\models^{\mathcal{M}}_{\beta} \psi$, $\overline{\models}^{\mathcal{M}}_{\gamma} \varphi$, and $\overline{\models}^{\mathcal{M}}_{\delta} \psi$;

  (iii) there is a world α in $\mathcal{M}$ at which $\models^{\mathcal{M}}_{\alpha} \varphi$ and $\overline{\models}^{\mathcal{M}}_{\alpha} \psi$;

  (iv) there is a world α in $\mathcal{M}$ at which $\overline{\models}^{\mathcal{M}}_{\alpha} \varphi$ and $\models^{\mathcal{M}}_{\alpha} \psi$;

  (v) there is a world β in $\mathcal{M}$ at which $\models^{\mathcal{M}}_{\beta} \varphi$ and $\models^{\mathcal{M}}_{\beta} \psi$.

(b) every atomic sentence of $\mathscr{L}$ is two-valued in $\mathcal{M}$.

Let χ be a sentence built using the primitives connectives of $\mathscr{L}$ except '|'. If χ does not contain any occurrence of atomic sentences, then either χ is $\mathcal{M}$–equivalent to '$\bot$' or χ is $\mathcal{M}$–equivalent to '$\neg\bot$'. In fact, the truth conditions of χ depend on no atomic sentence, so that χ has either no truth-value at every world or the same truth-value at all worlds. Moreover, by definition 2.1 χ is built using recursively the clauses (b)-(d) of such definition. By definition 4.5 χ is an ordinary sentence and therefore by theorem 4.5 χ is an essentially two valued sentence and *a fortiori* a two-valued sentence in $\mathcal{M}$. So, χ cannot be neither true nor false at every world in $\mathcal{M}$. In the case χ is $\mathcal{M}$–equivalent to '$\bot$', by condition (v) it holds that there is a world α in $\mathcal{M}$ such that $\models^{\mathcal{M}}_{\alpha} (\psi \mid \varphi)$, so that χ is not $\mathcal{M}$-equivalent to $(\psi \mid \varphi)$. In the case χ is $\mathcal{M}$–equivalent to '$\neg\bot$', by condition (iii) there is a world α in $\mathcal{M}$ at which it holds that $\overline{\models}^{\mathcal{M}}_{\alpha} (\psi \mid \varphi)$, so that, again, χ is not $\mathcal{M}$-equivalent to $(\psi \mid \varphi)$. Suppose now that at least one atomic sentence occurs in χ. Let $\{\omega_1, \ldots, \omega_n\} (1 \leq n)$ be the set of the atomic sentences occurring in χ. Since every atomic sentence ω_i $(1 \leq i \leq n)$ is, by conditions (b), two-valued in $\mathcal{M}$, it holds that $\uparrow\omega_i$ is $\mathcal{M}$–equivalent to ω_i. By definition 4.5 for every $i (1 \leq i \leq n)$ $\uparrow\omega_i$ is an ordinary sentence. By the same definition 4.5, χ is $\mathcal{M}$–equivalent to the sentence χ' obtained replacing in χ every occurrence of every $\uparrow\omega_i$ with ω_i. By theorem 4.5 χ' is essentially two-valued and χ, being $\mathcal{M}$–equivalent to χ', is two-valued in $\mathcal{M}$. Now, $(\psi \mid \varphi)$ is not two-valued, since by conditions (ii) and (iv) and 2.4 there is a world α at which neither $\models^{\mathcal{M}}_{\alpha} (\psi \mid \varphi)$ nor $\overline{\models}^{\mathcal{M}}_{\alpha} (\psi \mid \varphi)$. This proves that no sentence of $\mathscr{L}$ that do not contain occurrences of the symbol '|' may be $\mathcal{M}$–equivalent to $(\psi \mid \varphi)$.

q.e.d.

Ernest Adams developed (1975) a beautiful logic (APL) for these conditionals, albeit restricted to so-called *simple conditionals*. As anticipated above in section 1., our theory aims at extending Adams' theory to compounds of conditionals, providing for them a truth-conditional semantics. Since the present theory does not assume that conditionals are two-valued sentences, our task is not at odds with LTR. However, and we consider this is a very fundamental result, each compound of tri-events is logically equivalent to a simple tri-event (in the sense that they have the same truth conditions according to our semantics). Moreover, there is an effective way to associate a simple logically equivalent conditional to every tri-event.

Let us begin by defining the notion of simple tri-events properly. There are two definitions of this notion, one syntactical the other being semantical.

Definition 5.1 (Syntactically simple tri-events)

A sentence φ of $\mathscr{L}$ is said to be a syntactically simple tri-event iff $\varphi = (\psi \mid \chi)$ where ψ and χ are ordinary sentences.

Definition 5.2 (Semantically simple tri-events)

A sentence φ of $\mathscr{L}$ is said to be a semantically simple tri-event iff $\varphi = (\psi \mid \chi)$ and both ψ and χ are essentially two-valued sentences.

The following result, originally proved by de Finetti with respect to his truth-functional semantics (1995), which is kept in the present theory, shows that every sentence is logically equivalent to a semantically simple tri-event.

Theorem 5.4

Every sentence φ is logically equivalent to a semantically simple tri-event.

Proof

Let ψ be the sentence $(\uparrow\varphi \mid \updownarrow \varphi)$. We'll prove (a) that ψ is a semantically simple tri-event, and (b) that φ and ψ are logically equivalent.

(A) It suffices to prove that $\uparrow\varphi$ and $\updownarrow \varphi$ are ordinary sentences. $\uparrow\varphi$ is by definition 4.5 an ordinary sentence. $\updownarrow \varphi$ is by 2.2 an abbreviation of $(\uparrow\varphi \vee \uparrow\neg\varphi)$ that is the disjunction of two sentence that, by definition 4.5 are ordinary sentences and is therefore, by the same 4.5 an ordinary sentence.

(B) We prove (i) that $\{\varphi\} \models \psi$, and (ii) $\{\psi\} \models \varphi$.

(i) Suppose that in a model $\mathcal{M}$ and at the world α in $\mathcal{M}$ it holds that if φ. Then, by definition 2.4, $\models^{\mathcal{M}}_{\alpha} \uparrow\varphi$ and by theorem 2.1, $\models^{\mathcal{M}}_{\alpha} \updownarrow \varphi$. By definition 2.4, $\models^{\mathcal{M}}_{\alpha} (\uparrow\varphi \mid \updownarrow \varphi)$. Suppose now that not $\overline{\models}^{\mathcal{M}}_{\alpha}\ \varphi$ then, by theorem 2.1, either $\models^{\mathcal{M}}_{\alpha} \updownarrow \varphi$ and $\models^{\mathcal{M}}_{\alpha} \uparrow\varphi$ (so that as above $\models^{\mathcal{M}}_{\alpha} (\uparrow\varphi) \mid \updownarrow \varphi))$ or $\overline{\models}^{\mathcal{M}}_{\alpha} \updownarrow \varphi$, in which case, by definition 2.4, not $\overline{\models}^{\mathcal{M}}_{\alpha} (\uparrow\varphi \mid \updownarrow \varphi)$.

(ii) Suppose that at the world α in the model $\mathcal{M}$ it holds that $\models^{\mathcal{M}}_{\alpha} \psi$. Then $\models^{\mathcal{M}}_{\alpha} (\uparrow\varphi \mid\updownarrow \varphi)$. By definition 2.4 either both $\models^{\mathcal{M}}_{\alpha} \uparrow\varphi$ and $\models^{\mathcal{M}}_{\alpha}\updownarrow \varphi$ or, by theorem 3.2, ψ is valid. Now, if $\models^{\mathcal{M}}_{\alpha} \uparrow\varphi$ then $\models^{\mathcal{M}}_{\alpha} \varphi$. If $\models_{\mathcal{M}} \psi$ then it holds that $\models^{\mathcal{M}}_{\alpha} \psi$ at every world α. In this case, at any world α cannot be $\overline{\models}^{\mathcal{M}}_{\alpha} \varphi$ because, if so, would be that $\overline{\models}^{\mathcal{M}}_{\alpha} \uparrow\varphi$, being $\uparrow\varphi$ either true or false and not true by definition 2.4. In this case would be that $\models^{\mathcal{M}}_{\alpha}\updownarrow \varphi$, so that would be that $\overline{\models}^{\mathcal{M}}_{\alpha}$ $(\uparrow\varphi \mid\updownarrow \varphi)$. But $\psi = \uparrow(\varphi \mid\updownarrow \varphi)$, so would be that $\overline{\models}^{\mathcal{M}}_{\alpha} \psi$, contrary the assumption that $\models_{\mathcal{M}} \psi$. Moreover, if $\models_{\mathcal{M}} \psi$ there is a world α in $\mathcal{M}$ at which it holds that $\models^{\mathcal{M}}_{\alpha} \varphi$. Otherwise it would be that $\overline{\models}^{\mathcal{M}}_{\alpha} \uparrow\varphi$ at every world α in $\mathcal{M}$ and $\models^{\mathcal{M}}_{\alpha} \psi$ at no world in $\mathcal{M}$.
Suppose that at the world α in the model $\mathcal{M}$ it holds that not $\overline{\models}^{\mathcal{M}}_{\alpha} \psi$. We have already proved that if $\models^{\mathcal{M}}_{\alpha} \psi$ then also $\models^{\mathcal{M}}_{\alpha} \varphi$. Suppose then that ψ is such that not$\models^{\mathcal{M}}_{\alpha} \psi$ and not $\overline{\models}^{\mathcal{M}}_{\alpha} \psi$. In this case $\overline{\models}^{\mathcal{M}}_{\alpha}\updownarrow \varphi$ and by definition 2.4, not $\models^{\mathcal{M}}_{\alpha} (\uparrow\varphi \mid\updownarrow \varphi)$ and not $\overline{\models}^{\mathcal{M}}_{\alpha} (\uparrow\varphi \mid\updownarrow \varphi)$, in which case would be not $\models^{\mathcal{M}}_{\alpha} \psi$ and not $\overline{\models}^{\mathcal{M}}_{\alpha} \psi$.

q.e.d.

Before proving the syntactical counterpart of theorem 5.4, we will prove some relevant results from which such a theorem follows quickly. Some of the following equivalences are based on the fact that the truth conditions that govern connectives may be made explicit by means of modal symbols and the unary symbol '$\uparrow$', along the other connectives.

Theorem 5.5

The following equivalences hold:

1. $\models \uparrow(\varphi \vee \psi) \leftrightarrow ((\uparrow\varphi \vee \uparrow\psi) \vee (\neg\Diamond(\uparrow\neg\varphi \wedge \uparrow\neg\psi) \wedge (\Diamond\uparrow\varphi \vee \Diamond\uparrow\psi)))$
2. $\models \uparrow\neg(\varphi \vee \psi) \leftrightarrow (\uparrow\neg\varphi \wedge \uparrow\neg\psi)$
3. $\models \uparrow(\varphi \wedge \psi) \leftrightarrow (\uparrow\varphi \wedge \uparrow\psi)$
4. $\models \uparrow\neg(\varphi \wedge \psi) \leftrightarrow (\uparrow\neg\varphi \vee \uparrow\neg\psi \vee (\neg\Diamond(\uparrow\varphi \wedge \uparrow\psi) \wedge (\Diamond\uparrow\neg\varphi \vee \Diamond\uparrow\neg\psi)))$
5. $\models \uparrow(\psi \mid \varphi) \leftrightarrow ((\uparrow\varphi \wedge \uparrow\psi) \vee (\Diamond(\uparrow\varphi \wedge \uparrow\psi) \wedge \neg\Diamond(\uparrow\varphi \wedge \uparrow\neg\psi)))$
6. $\models \uparrow\neg(\psi \mid \varphi) \leftrightarrow \uparrow(\neg\psi \mid \varphi)$
7. $\models \uparrow\Diamond\varphi \leftrightarrow \Diamond\uparrow\varphi$
8. $\models \uparrow\neg\Diamond\varphi \leftrightarrow (\uparrow(\varphi \vee \neg\varphi) \wedge \neg\Diamond\uparrow\varphi)$
9. $\models \uparrow(\varphi \to \psi) \leftrightarrow (\uparrow\psi \vee (\neg\uparrow\varphi \wedge \neg\uparrow\neg\psi)) \vee \uparrow\neg\varphi)$
10. $\models \uparrow\neg(\varphi \to \psi) \leftrightarrow ((\uparrow\varphi \wedge \neg\uparrow\psi) \vee (\uparrow\neg\psi \wedge \neg\uparrow\neg\varphi))$

Proof

By definition 2.4 for every pair of sentences φ and ψ, every model $\mathcal{M}$ and every world α in $\mathcal{M}$, it holds that (i) $\models^{\mathcal{M}}_{\alpha} \uparrow\varphi$ iff $\models^{\mathcal{M}}_{\alpha} \varphi$, and that (ii) $\models_{\mathcal{M}} (\varphi \leftrightarrow$

ψ) amounts to the satisfaction of the two following conditions: (i) $\models^{\mathcal{M}}_{\alpha} \varphi$ iff $\models^{\mathcal{M}}_{\alpha} \psi$ and (ii) $\overline{\models}^{\mathcal{M}}_{\alpha} \varphi$ iff $\overline{\models}^{\mathcal{M}}_{\alpha} \psi$. Since in all the equivalences both the left side and the right side are ordinary sentences by definition 4.5, only the condition (i) needs to be proved. We prove:

1. $\models \uparrow(\varphi \vee \psi) \leftrightarrow ((\uparrow\varphi \vee \uparrow\psi) \vee (\neg\Diamond(\uparrow\neg\varphi \wedge \uparrow\neg\psi) \wedge (\Diamond\uparrow\varphi \vee \Diamond\uparrow\psi)))$

    (a) Suppose, first, that for some model $\mathcal{M}$ and some world α in $\mathcal{M}$, it holds that $\models^{\mathcal{M}}_{\alpha} (\varphi \vee \psi)$. We have to prove $\models^{\mathcal{M}}_{\alpha} ((\uparrow\varphi \vee \uparrow\psi) \vee (\neg\Diamond(\uparrow\neg\varphi \wedge \uparrow\neg\psi) \wedge (\Diamond\uparrow\varphi \vee \Diamond\uparrow\psi)))$. Suppose, *ab absurdo*, that this last sentence is false at some world α in $\mathcal{M}$. Since both $\chi = (\uparrow\varphi \vee \uparrow\psi)$ and $\zeta = (\neg\Diamond\uparrow\neg\varphi \wedge \uparrow\neg\psi) \wedge (\Diamond\uparrow\varphi \vee \Diamond\uparrow\psi)))$ are ordinary sentences, the entire sentence may be false only if holds that $\overline{\models}^{\mathcal{M}}_{\alpha} \chi$ and $\overline{\models}^{\mathcal{M}}_{\alpha} \zeta$. Now, $\overline{\models}^{\mathcal{M}}_{\alpha} \chi$ only if both $\overline{\models}^{\mathcal{M}}_{\alpha} \varphi$ and $\overline{\models}^{\mathcal{M}}_{\alpha} \psi$. $\overline{\models}^{\mathcal{M}}_{\alpha} \zeta$ only if at least one of its conjuncts is false at α. The first conjunct $(\neg\Diamond\uparrow\neg\varphi \wedge \uparrow\neg\psi)$ is false at α only if there is a world β in $\mathcal{M}$ at which both $\overline{\models}^{\mathcal{M}}_{\beta} \varphi$ and $\overline{\models}^{\mathcal{M}}_{\beta} \psi$ or not $\models^{\mathcal{M}}_{\gamma} \varphi$ and not $\models^{\mathcal{M}}_{\gamma} \psi$ at every world γ in $\mathcal{M}$. But in such a case, by definition 2.4, $\overline{\models}^{\mathcal{M}}_{\alpha} \uparrow(\varphi \vee \psi)$, which contradicts our hypothesis.

    (b) Suppose now that $((\uparrow\varphi\vee\uparrow\psi)\vee(\neg\Diamond(\uparrow\neg\varphi\wedge\uparrow\neg\psi)\wedge(\Diamond\uparrow\varphi\vee\Diamond\uparrow\psi)))$ is true at some world α in a certain model $\mathcal{M}$. So either $\chi = (\uparrow\varphi\vee\uparrow\psi)$ is true at α or $\zeta = (\neg\Diamond(\uparrow\neg\varphi\wedge\uparrow\neg\psi)\wedge(\Diamond\uparrow\varphi\vee\Diamond\uparrow\psi)))$ is true at α. In the first case, either $\models^{\mathcal{M}}_{\alpha} \varphi$ or $\models^{\mathcal{M}}_{\alpha} \psi$, so that, by definition 2.4, also $\models^{\mathcal{M}}_{\alpha} \uparrow(\varphi \vee \psi)$. In the second case, there is no world β in $\mathcal{M}$ such that $\overline{\models}^{\mathcal{M}}_{\beta} \varphi$ and $\overline{\models}^{\mathcal{M}}_{\beta} \psi$ and moreover there is a world γ at which either $\models^{\mathcal{M}}_{\gamma} \varphi$ or $\models^{\mathcal{M}}_{\gamma} \psi$. By definition 2.4, again, $\models^{\mathcal{M}}_{\alpha} \uparrow(\varphi \vee \psi)$.

2. $\models \uparrow\neg(\varphi \vee \psi) \leftrightarrow (\uparrow\neg\varphi \wedge \uparrow\neg\psi)$

    (a) Suppose that for some model $\mathcal{M}$ and some world α in $\mathcal{M}$, it holds that $\models^{\mathcal{M}}_{\alpha} \uparrow\neg(\varphi\vee\psi)$. By definition 2.4 it holds that $\overline{\models}^{\mathcal{M}}_{\alpha} (\varphi\vee\psi)$. By definition 2.4 it holds also that (j) either $\overline{\models}^{\mathcal{M}}_{\alpha} \varphi$ and $\overline{\models}^{\mathcal{M}}_{\alpha} \psi$ or (jj) for no world β in $\mathcal{M}$ it holds that either $\models^{\mathcal{M}}_{\beta} \varphi$ or $\models^{\mathcal{M}}_{\beta} \psi$ and there is a world γ in $\mathcal{M}$ such that it holds that $\overline{\models}^{\mathcal{M}}_{\gamma} \varphi$ and $\overline{\models}^{\mathcal{M}}_{\gamma} \psi$. In the case (j)it holds soon that $(\uparrow\neg\varphi \wedge \uparrow\neg\psi)$. In the case (jj) $(\varphi \vee \psi)$ is countervalid in $\mathcal{M}$ so that at every world γ in $\mathcal{M}$ it holds that $\overline{\models}^{\mathcal{M}}_{\gamma} \varphi$ and $\overline{\models}^{\mathcal{M}}_{\gamma} \psi$ and also $\models^{\mathcal{M}}_{\gamma} \uparrow\neg\varphi$ and $\models^{\mathcal{M}}_{\gamma} \uparrow\neg\psi$, from which it follows that $\models^{\mathcal{M}}_{\alpha} (\uparrow\neg\varphi \wedge \uparrow\neg\psi)$.

    (b) Suppose that for some model $\mathcal{M}$ and some world α in $\mathcal{M}$, it holds that $\models^{\mathcal{M}}_{\alpha} (\uparrow\neg\varphi \wedge \uparrow\neg\psi)$. By definition 2.4 $\overline{\models}^{\mathcal{M}}_{\alpha} \varphi$ and $\overline{\models}^{\mathcal{M}}_{\alpha} \psi$ and also $\overline{\models}^{\mathcal{M}}_{\alpha} (\varphi \vee \psi)$ and therefore $\models^{\mathcal{M}}_{\alpha} \uparrow\neg(\varphi \vee \psi)$.

3. $\models \uparrow(\varphi \wedge \psi) \leftrightarrow (\uparrow\varphi \wedge \uparrow\psi)$

(a) Suppose that for some model $\mathcal{M}$ and some world α in $\mathcal{M}$, it holds that $\models^{\mathcal{M}}_{\alpha} \{\uparrow(\varphi \wedge \psi)$. Suppose, *ab absurdo*, that not $\models^{\mathcal{M}}_{\alpha} (\uparrow\varphi \wedge \uparrow\psi)$. Since $(\uparrow\varphi \wedge \uparrow\psi)$ is an ordinary sentence, it follows that $\overline{\models}^{\mathcal{M}}_{\alpha} (\uparrow\varphi \wedge \uparrow\psi)$. Hence either $\overline{\models}^{\mathcal{M}}_{\alpha} \varphi$ or $\overline{\models}^{\mathcal{M}}_{\alpha} \psi$. In such a case $\overline{\models}^{\mathcal{M}}_{\alpha} (\varphi \wedge \psi)$ and therefore $\overline{\models}^{\mathcal{M}}_{\alpha} \uparrow(\varphi \wedge \psi)$, against the hypothesis.

(b) Suppose that $\models^{\mathcal{M}}_{\alpha} (\uparrow\varphi \wedge \uparrow\psi)$. Then $\models^{\mathcal{M}}_{\alpha} \varphi$ and $\models^{\mathcal{M}}_{\alpha} \psi$. By definition 2.4, $\models^{\mathcal{M}}_{\alpha} (\varphi \wedge \psi)$ and $\models^{\mathcal{M}}_{\alpha} \uparrow(\varphi \wedge \psi)$.

4. $\models \uparrow\neg(\varphi \wedge \psi) \leftrightarrow (\uparrow\neg\varphi \vee \uparrow\neg\psi \vee (\neg\Diamond(\uparrow\varphi \wedge \uparrow\psi) \wedge (\Diamond\uparrow\neg\varphi \vee \Diamond\uparrow\neg\psi)))$

Let ζ be $(\uparrow\neg\varphi \vee \uparrow\neg\psi \vee (\neg\Diamond(\uparrow\varphi \wedge \uparrow\psi) \wedge (\Diamond\uparrow\neg\varphi \vee \Diamond\uparrow\neg\psi)))$.

(a) Suppose that for some model $\mathcal{M}$ and some world α in $\mathcal{M}$, it holds that $\models^{\mathcal{M}}_{\alpha} \uparrow\neg(\varphi \wedge \psi)$. By definition 2.4 it holds that $\overline{\models}^{\mathcal{M}}_{\alpha} (\varphi \wedge \psi)$. By definition 2.4 it holds that either (j) $\overline{\models}^{\mathcal{M}}_{\alpha} \varphi$ or $\overline{\models}^{\mathcal{M}}_{\alpha} \psi$ or (jj) for no world β in $\mathcal{M}$ it holds that $\models^{\mathcal{M}}_{\alpha} \varphi$ and $\models^{\mathcal{M}}_{\alpha} \psi$ and at least one of the following conditions is satisfied: (k) there exists γ in $\mathcal{M}$ such that $\overline{\models}^{\mathcal{M}}_{\gamma} \varphi$, (kk) there exists γ in $\mathcal{M}$ such that $\overline{\models}^{\mathcal{M}}_{\gamma} \psi$. In the case (j) it holds, by definition 2.4, that $\overline{\models}^{\mathcal{M}}_{\alpha} (\varphi \wedge \psi)$ and also that $\models^{\mathcal{M}}_{\alpha} (\uparrow\neg\varphi \vee \uparrow\neg\psi)$ and also ζ. In the case (jj), at very world β $\overline{\models}^{\mathcal{M}}_{\beta} (\varphi \wedge \psi)$ so that, by definition 2.4 $\models^{\mathcal{M}}_{\alpha} \neg\Diamond(\uparrow\varphi \wedge \uparrow\psi)$. In this case it holds that $\models^{\mathcal{M}}_{\alpha} \zeta$.

(b) Suppose that for some model $\mathcal{M}$ and some world α in $\mathcal{M}$, it holds that $\models^{\mathcal{M}}_{\alpha} \zeta$. ζ is a disjunction of sub-sentences that, by definition 2.4 are either true or false at α. Therefore, either (j) $\models^{\mathcal{M}}_{\alpha} \uparrow\neg\varphi$ or (jj) $\models^{\mathcal{M}}_{\alpha} \uparrow\neg\psi$ or (jjj) $\models^{\mathcal{M}}_{\alpha} (\neg\Diamond(\uparrow\varphi \wedge \uparrow\psi) \wedge (\Diamond\uparrow\neg\varphi \vee \Diamond\uparrow\neg\psi))$. In the case (j) and (jj), ir follows from definition 2.4 that $\models^{\mathcal{M}}_{\alpha} \uparrow\neg(\varphi \wedge \psi)$. In the case (jjj) for every world β in $\mathcal{M}$ it holds that $\overline{\models}^{\mathcal{M}}_{\beta} (\varphi \wedge \psi)$ and therefore that $\overline{\models}^{\mathcal{M}}_{\alpha} (\varphi \wedge \psi)$. From definition 2.4 it follows that $\models^{\mathcal{M}}_{\alpha} \uparrow\neg(\varphi \wedge \psi)$.

5. $\models \uparrow(\psi \mid \varphi) \leftrightarrow ((\uparrow\varphi \wedge \uparrow\psi) \vee (\Diamond(\uparrow\varphi \wedge \uparrow\psi) \wedge \neg\Diamond(\uparrow\varphi \wedge \uparrow\neg\psi)))$

(a) Suppose, that for some model $\mathcal{M}$ and some world α in $\mathcal{M}$, it holds that $\models^{\mathcal{M}}_{\alpha} \uparrow(\psi \mid \varphi)$. By definition 2.4, either (j) both $\models^{\mathcal{M}}_{\alpha} \varphi$ and $\models^{\mathcal{M}}_{\alpha} \psi$ or (jj) there is a world β such that $\models^{\mathcal{M}}_{\beta} \varphi$ and $\models^{\mathcal{M}}_{\beta} \psi$ but there is no world γ such that $\models^{\mathcal{M}}_{\gamma} \varphi$ and $\overline{\models}^{\mathcal{M}}_{\gamma} \psi$. Now if (j) is satisfied then it holds that $\models^{\mathcal{M}}_{\alpha} (\uparrow\varphi \wedge \uparrow\psi)$, so that $\models^{\mathcal{M}}_{\alpha} \Diamond(\uparrow\varphi \wedge \uparrow\psi)$. If (jj) is satisfied then it holds that $\models^{\mathcal{M}}_{\beta} (\uparrow\varphi \wedge \uparrow\psi)$ for some β and for every γ it holds that not $\models^{\mathcal{M}}_{\gamma} (\uparrow\varphi \wedge \uparrow\neg\psi)$, so that $\models^{\mathcal{M}}_{\alpha} \neg\Diamond(\uparrow\varphi \wedge \uparrow\neg\psi)$. It follows that $\models^{\mathcal{M}}_{\alpha} ((\uparrow\varphi \wedge \uparrow\psi) \vee (\Diamond(\uparrow\varphi \wedge \uparrow\psi) \wedge \neg\Diamond(\uparrow\varphi \wedge \uparrow\neg\psi)))$ as desired.

(b) If $\models^{\mathcal{M}}_{\alpha} ((\uparrow\varphi \wedge \uparrow\psi) \vee (\Diamond(\uparrow\varphi \wedge \uparrow\psi) \wedge \neg\Diamond(\uparrow\varphi \wedge \uparrow\neg\psi)))$, since all subformulas of this sentence are ordinary sentences either (k) $\models^{\mathcal{M}}_{\alpha} ((\uparrow\varphi \wedge \uparrow\psi)$ or (kk) or $\models^{\mathcal{M}}_{\alpha} (\Diamond(\uparrow\varphi \wedge \uparrow\psi) \wedge \neg\Diamond(\uparrow\varphi \wedge \uparrow\neg\psi))$ In the case (k), by definition 2.4, $\models^{\mathcal{M}}_{\alpha} \uparrow(\psi \mid \varphi)$. In the case (kk), it holds that $\models^{\mathcal{M}}_{\alpha} \Diamond(\uparrow\varphi \wedge \uparrow\psi)$ and $\models^{\mathcal{M}}_{\alpha} \neg\Diamond(\uparrow\varphi \wedge \uparrow\neg\psi)$ and hence, by definition 2.4, $\models^{\mathcal{M}}_{\alpha} \uparrow(\psi \mid \varphi)$.

6. $\models \uparrow\neg(\psi \mid \varphi) \leftrightarrow \uparrow(\neg\psi \mid \varphi)$

(a) Suppose that for some model $\mathcal{M}$ and some world α in $\mathcal{M}$, it holds that $\models^{\mathcal{M}}_{\alpha} \uparrow\neg(\psi \mid \varphi)$. By definition 2.4 it holds that $\overline{\models}^{\mathcal{M}}_{\alpha} (\psi \mid \varphi)$. But this entails that either (a) $\models^{\mathcal{M}}_{\alpha} \varphi$ and $\overline{\models}^{\mathcal{M}}_{\alpha} \psi$ or (b) there is a world β in $\mathcal{M}$ such that $\models^{\mathcal{M}}_{\beta} \varphi$ and $\overline{\models}^{\mathcal{M}}_{\beta} \psi$, and for every world γ in $\mathcal{M}$ not $\models^{\mathcal{M}}_{\gamma} \varphi$ and not $\models^{\mathcal{M}}_{\gamma} \psi$. In the case (a) $\models^{\mathcal{M}}_{\alpha} \neg\psi$, so that $\models^{\mathcal{M}}_{\alpha} \uparrow(\neg\psi \mid \varphi)$. In the case (b) $\models^{\mathcal{M}}_{\beta} \neg\psi$ not $\overline{\models}^{\mathcal{M}}_{\gamma} \neg\psi$, so that, by definition 2.4 $\models^{\mathcal{M}}_{\alpha} (\psi \mid \varphi)$.

(b) Suppose that for some model $\mathcal{M}$ and some world α in $\mathcal{M}$, it holds that $\models^{\mathcal{M}}_{\alpha} \uparrow(\neg\psi \mid \varphi)$. By point 3 above, it holds that $\models^{\mathcal{M}}_{\alpha} ((\uparrow\varphi \wedge \uparrow\neg\psi) \vee (\Diamond(\uparrow\varphi \wedge \uparrow\neg\psi) \wedge \neg\Diamond(\uparrow\varphi \wedge \uparrow\neg\neg\psi)))$. Either $\models^{\mathcal{M}}_{\alpha} (\uparrow\varphi \wedge \uparrow\neg\psi)$ or $\models^{\mathcal{M}}_{\alpha} (\Diamond(\uparrow\varphi \wedge \uparrow\neg\psi) \wedge \neg\Diamond(\uparrow\varphi \wedge \uparrow\neg\neg\psi))$. In the first case, by definition 2.4 $\models^{\mathcal{M}}_{\alpha} \uparrow\neg(\psi \mid \varphi)$. In the second case, there is a world β in $\mathcal{M}$ such that $\models^{\mathcal{M}}_{\beta} (\uparrow\varphi \wedge \uparrow\neg\psi)$ and for no world γ in $\mathcal{M}$ it holds that $\models^{\mathcal{M}}_{\gamma} (\uparrow\varphi \wedge \uparrow\neg\neg\psi))$. Considering that by theorem 5.2 $\neg\neg\psi$ is logically equivalent to ψ there is no world γ in $\mathcal{M}$ at which $\models^{\mathcal{M}}_{\gamma} (\uparrow\varphi \wedge \uparrow\psi)$. By definition 2.4 it holds that $\overline{\models}^{\mathcal{M}}_{\alpha} \uparrow(\psi \mid \varphi)$ so that, by theorem 5.1, $\models^{\mathcal{M}}_{\alpha} \uparrow\neg(\psi \mid \varphi)$

7. $\models \uparrow\Diamond\varphi \leftrightarrow \Diamond\uparrow\varphi$

(a) Suppose that for some model $\mathcal{M}$ and some world α in $\mathcal{M}$, it holds that $\models^{\mathcal{M}}_{\alpha} \uparrow\Diamond\varphi$. By definition 2.4, it follows that $\models^{\mathcal{M}}_{\alpha} \Diamond\varphi$. By definition 2.4, there is a world β such that $\models^{\mathcal{M}}_{\beta} \varphi$ and also $\models^{\mathcal{M}}_{\beta} \uparrow\varphi$. It follows from definition 2.4 that $\models^{\mathcal{M}}_{\alpha} \Diamond\uparrow\varphi$.

(b) Suppose that for some model $\mathcal{M}$ and some world α in $\mathcal{M}$, it holds that $\models^{\mathcal{M}}_{\alpha} \Diamond\uparrow\varphi$. By definition 2.4, there is a world β such that $\models^{\mathcal{M}}_{\beta} \uparrow\varphi$ and also $\models^{\mathcal{M}}_{\beta} \varphi$. It follows $\models^{\mathcal{M}}_{\alpha} \Diamond\varphi$ and also $\models^{\mathcal{M}}_{\alpha} \uparrow\Diamond\varphi$

8. $\models \uparrow\neg\Diamond\varphi \leftrightarrow (\uparrow(\varphi \vee \neg\varphi) \wedge \neg\Diamond\uparrow\varphi)$

(a) Suppose that for some model $\mathcal{M}$ and some world α in $\mathcal{M}$, it holds that $\models^{\mathcal{M}}_{\alpha} \uparrow\neg\Diamond\varphi$. By definition 2.4, it holds that $\overline{\models}^{\mathcal{M}}_{\alpha} \Diamond\varphi$, so that $\overline{\models}_{\mathcal{M}} \varphi$. This result, in turn, entails that φ is two-valued. Hence $\uparrow(\varphi \vee \neg\varphi)$. Moreover, $\models^{\mathcal{M}}_{\alpha} \neg\Diamond\uparrow\varphi$. By definition 2.4 it holds that $\models^{\mathcal{M}}_{\alpha} (\uparrow(\varphi \vee \neg\varphi) \wedge \neg\Diamond\uparrow\varphi)$.

(b) Suppose that for some model $\mathcal{M}$ and some world α in $\mathcal{M}$, it holds that $\models^{\mathcal{M}}_{\alpha} (\uparrow(\varphi \vee \neg\varphi) \wedge \neg\Diamond\uparrow\varphi)$. It follows that $\models^{\mathcal{M}}_{\alpha} \uparrow(\varphi \vee \neg\varphi)$ and $\models^{\mathcal{M}}_{\alpha} \neg\Diamond\uparrow\varphi$. From $\models^{\mathcal{M}}_{\alpha} \uparrow(\varphi \vee \neg\varphi)$ it follows that either $\models^{\mathcal{M}}_{\alpha} \varphi$ or $\overline{\models}^{\mathcal{M}}_{\alpha} \varphi$ and from that and $\models^{\mathcal{M}}_{\alpha} \neg\Diamond\uparrow\varphi$ it follows that $\overline{\models}^{\mathcal{M}}_{\alpha} \varphi$. Since this holds for every world β in $\mathcal{M}$, it holds that $\models^{\mathcal{M}}_{\alpha} \uparrow\neg\Diamond\varphi$.

9. $\models \uparrow(\varphi \rightarrow \psi) \leftrightarrow (\uparrow\psi \vee (\neg\uparrow\varphi \wedge \neg\uparrow\neg\psi) \vee \uparrow\neg\varphi)$
Let ζ be $(\uparrow\psi \vee (\neg\uparrow\varphi \wedge \neg\uparrow\neg\psi) \vee \uparrow\neg\varphi)$.

(a) Suppose that for some model $\mathcal{M}$ and some world α in $\mathcal{M}$, it holds that $\models^{\mathcal{M}}_{\alpha} \uparrow(\varphi \rightarrow \psi)$. This entails by definition 2.4 that $\models^{\mathcal{M}}_{\alpha} (\varphi \rightarrow \psi)$. This, in turn, entails that either $\models^{\mathcal{M}}_{\alpha} \psi$ or $\overline{\models}^{\mathcal{M}}_{\alpha} \varphi$ or both not $\models^{\mathcal{M}}_{\alpha} \varphi$ and not $\overline{\models}^{\mathcal{M}}_{\alpha} \psi$. If $\models^{\mathcal{M}}_{\alpha} \psi$ then $\models^{\mathcal{M}}_{\alpha} \uparrow\psi$, so that it also holds that $\models^{\mathcal{M}}_{\alpha} \zeta$. If $\overline{\models}^{\mathcal{M}}_{\alpha} \varphi$ then $\models^{\mathcal{M}}_{\alpha} \neg\varphi$ which entails $\models^{\mathcal{M}}_{\alpha} \uparrow\neg\varphi$, so that so that it also holds that $\models^{\mathcal{M}}_{\alpha} \zeta$. If neither $\models^{\mathcal{M}}_{\alpha} \varphi$ nor $\overline{\models}^{\mathcal{M}}_{\alpha} \psi$ it holds that $\models^{\mathcal{M}}_{\alpha} (\neg\uparrow\varphi \wedge \neg\uparrow\neg\psi)$, so that so that it also holds that $\models^{\mathcal{M}}_{\alpha} \zeta$.

(b) Suppose that for some model $\mathcal{M}$ and some world α in $\mathcal{M}$, it holds that $\models^{\mathcal{M}}_{\alpha} \zeta$. This entails, by definition 2.4 that either $\models^{\mathcal{M}}_{\alpha} \psi$ or $\overline{\models}^{\mathcal{M}}_{\alpha} \varphi$ or neither $\models^{\mathcal{M}}_{\alpha} \varphi$ nor $\overline{\models}^{\mathcal{M}}_{\alpha} \psi$. By definition 2.4, each of these conditions entail that $\models^{\mathcal{M}}_{\alpha} (\varphi \rightarrow \psi)$ and also $\models^{\mathcal{M}}_{\alpha} \uparrow(\varphi \rightarrow \psi)$.

10. $\models\uparrow\neg(\varphi \rightarrow \psi) \leftrightarrow ((\uparrow\varphi \wedge \neg\uparrow\psi) \vee (\uparrow\neg\psi \wedge \neg\uparrow\neg\varphi))$
Let ζ be $((\uparrow\varphi \wedge \neg\uparrow\psi) \vee (\uparrow\neg\psi \wedge \neg\uparrow\neg\varphi))$.

(a) Suppose that for some model $\mathcal{M}$ and some world α in $\mathcal{M}$, it holds that $\models^{\mathcal{M}}_{\alpha} \uparrow\neg(\varphi \rightarrow \psi)$. It follows that $\overline{\models}^{\mathcal{M}}_{\alpha} (\varphi \rightarrow \psi)$. By definition 2.4, either $\models^{\mathcal{M}}_{\alpha} \varphi$ and not $\models^{\mathcal{M}}_{\alpha} \psi$ or $\overline{\models}^{\mathcal{M}}_{\alpha} \psi$ and not $\overline{\models}^{\mathcal{M}}_{\alpha} \varphi$. Now, if $\models^{\mathcal{M}}_{\alpha} \varphi$ then $\models^{\mathcal{M}}_{\alpha} \uparrow\varphi$ and if not $\models^{\mathcal{M}}_{\alpha} \psi$ then $\models^{\mathcal{M}}_{\alpha} \neg\uparrow\psi$, so that it holds that $\models^{\mathcal{M}}_{\alpha} (\uparrow\varphi \wedge \neg\uparrow\psi)$. On the other hand, if $\overline{\models}^{\mathcal{M}}_{\alpha} \psi$ then $\models^{\mathcal{M}}_{\alpha} \uparrow\neg\psi$ ad if not $\overline{\models}^{\mathcal{M}}_{\alpha} \varphi$ then $\models^{\mathcal{M}}_{\alpha} \neg\uparrow\neg\varphi$, so that it holds that $\models^{\mathcal{M}}_{\alpha} (\uparrow\neg\psi \wedge \neg\uparrow\neg\varphi)$. So also $\models^{\mathcal{M}}_{\alpha} \zeta$.

(b) Suppose that for some model $\mathcal{M}$ and some world α in $\mathcal{M}$, it holds that $\models^{\mathcal{M}}_{\alpha} \zeta$ so that either $\models^{\mathcal{M}}_{\alpha} (\uparrow\varphi \wedge \neg\uparrow\psi)$ or $\models^{\mathcal{M}}_{\alpha} (\uparrow\neg\psi \wedge \neg\uparrow\neg\varphi)$. In the first case it holds that $\models^{\mathcal{M}}_{\alpha} \varphi$ and not $\models^{\mathcal{M}}_{\alpha} \psi$. By definition 2.4, $\overline{\models}^{\mathcal{M}}_{\alpha} (\varphi \rightarrow \psi)$ and therefore $\models^{\mathcal{M}}_{\alpha} \uparrow\neg(\varphi \rightarrow \psi)$. In the second case, $\overline{\models}^{\mathcal{M}}_{\alpha} \psi$ and not $\overline{\models}^{\mathcal{M}}_{\alpha} \varphi$. By definition 2.4, $\overline{\models}^{\mathcal{M}}_{\alpha} (\varphi \rightarrow \psi)$ and therefore $\models^{\mathcal{M}}_{\alpha} \uparrow\neg(\varphi \rightarrow \psi)$. So, in every case it holds that $\models^{\mathcal{M}}_{\alpha} \uparrow\neg(\varphi \rightarrow \psi)$.

q.e.d.

Theorem 5.6

(De Morgan Laws). Let φ and ψ to sentences of Let φ be a sentence of $\mathscr{L}$. The following equivalences hold:

1. $\models (\varphi \wedge \psi) \leftrightarrow \neg(\neg\varphi \vee \neg\psi)$
2. $\models (\varphi \vee \psi) \leftrightarrow \neg(\neg\varphi \wedge \neg\psi)$
3. $\models \neg(\varphi \wedge \psi) \leftrightarrow (\neg\varphi \vee \neg\psi)$
4. $\models \neg(\varphi \vee \psi) \leftrightarrow (\neg\varphi \wedge \neg\psi)$

Proof

1. $\models (\varphi \wedge \psi) \leftrightarrow \neg(\neg\varphi \vee \neg\psi)$
   (a) Suppose that for a certain world α in the model $\mathcal{M}$ it holds that $\models^{\mathcal{M}}_{\alpha} (\varphi \wedge \psi)$. By definition 2.4 it holds that $\models^{\mathcal{M}}_{\alpha} \uparrow(\varphi \wedge \psi)$. By theorem 5.5 it holds that $\models^{\mathcal{M}}_{\alpha} (\uparrow\varphi \wedge \uparrow\psi)$ and therefore $\models^{\mathcal{M}}_{\alpha} \varphi$ and $\models^{\mathcal{M}}_{\alpha} \psi$. It follows that $\overline{\models}^{\mathcal{M}}_{\alpha} \neg\varphi)$ and $\overline{\models}^{\mathcal{M}}_{\alpha} \neg\psi$. By definition 2.4 it holds that $\overline{\models}^{\mathcal{M}}_{\alpha} (\neg\varphi \vee \neg\psi)$ and, finally, that $\models^{\mathcal{M}}_{\alpha} \neg(\neg\varphi \vee \neg\psi)$.

   (b) Suppose that for a certain world α in the model $\mathcal{M}$ it holds that $\models^{\mathcal{M}}_{\alpha} \neg(\neg\varphi \vee \neg\psi)$. It holds that $\models^{\mathcal{M}}_{\alpha} \uparrow\neg(\neg\varphi \vee \neg\psi)$. By theorem 5.5 it holds that $\models^{\mathcal{M}}_{\alpha} (\uparrow\neg\neg\varphi \wedge \uparrow\neg\neg\psi$. By theorem 5.2 it holds that $\models^{\mathcal{M}}_{\alpha} (\uparrow\varphi \wedge \uparrow\psi$. By theorem 5.5 it holds $\models^{\mathcal{M}}_{\alpha} \uparrow(\varphi \wedge \psi)$. Therefore, by definition 2.4 it holds that $\models^{\mathcal{M}}_{\alpha} (\varphi \wedge \psi)$.

   (c) Suppose that for a certain world α in the model $\mathcal{M}$ it holds that $\overline{\models}^{\mathcal{M}}_{\alpha} (\varphi \wedge \psi)$. By definition 2.4, there are three exhaustive cases: (j) $\models^{\mathcal{M}}_{\alpha} \uparrow\neg\varphi$, (jj) $\models^{\mathcal{M}}_{\alpha} \uparrow\neg\psi$, (jjj) it holds that either that (k) for every world β in $\mathcal{M}$, either not $\models^{\mathcal{M}}_{\beta} \varphi$ or not $\models^{\mathcal{M}}_{\beta} \psi$, or (kk) there is a world γ in $\mathcal{M}$ such that it holds that either $\overline{\models}^{\mathcal{M}}_{\gamma} \varphi$ or $\overline{\models}^{\mathcal{M}}_{\gamma} \psi$. In the case (j) it holds that $\overline{\models}^{\mathcal{M}}_{\alpha} \varphi$ and therefore $\models^{\mathcal{M}}_{\alpha} \neg\varphi$ and also $\models^{\mathcal{M}}_{\alpha} (\neg\varphi \vee \neg\psi)$ and therefore $\overline{\models}^{\mathcal{M}}_{\alpha} \neg(\neg\varphi \vee \neg\psi)$. In the case (jj) it holds that $\overline{\models}^{\mathcal{M}}_{\alpha} \psi$ and therefore $\models^{\mathcal{M}}_{\alpha} \neg\psi$ and also $\models^{\mathcal{M}}_{\alpha} (\neg\varphi \vee \neg\psi)$ and therefore $\overline{\models}^{\mathcal{M}}_{\alpha} \neg(\neg\varphi \vee \neg\psi)$. In the case (jjj) $\overline{\models}_{\mathcal{M}} (\varphi \wedge \psi)$ so that $\models_{\mathcal{M}} \neg(\varphi \wedge \psi)$. So at every world β in $\mathcal{M}$ it holds that either $\models^{\mathcal{M}}_{\beta} \neg\varphi$ or $\models^{\mathcal{M}}_{\beta} \neg\psi$. It follows that $\models^{\mathcal{M}}_{\beta} (\neg\varphi \vee \psi)$ and therefore that $\overline{\models}^{\mathcal{M}}_{\beta} \neg(\neg\varphi \vee \psi)$.

   (d) Suppose that for a certain world α in the model $\mathcal{M}$ it holds that $\overline{\models}^{\mathcal{M}}_{\alpha} \neg(\neg\varphi \vee \neg\psi)$. By theorem 5.5 it holds that $\models^{\mathcal{M}}_{\alpha} (\neg\varphi \vee \neg\psi)$. By definition 2.4, there are two exhaustive cases: (j) either $\overline{\models}^{\mathcal{M}}_{\alpha} \varphi$ or $\overline{\models}^{\mathcal{M}}_{\alpha} \psi$, and (jj) in no world β in $\mathcal{M}$ it holds that $\models^{\mathcal{M}}_{\beta} \varphi$ and $\models^{\mathcal{M}}_{\beta} \psi$ and there is a world γ in $\mathcal{M}$ such that either $\overline{\models}^{\mathcal{M}}_{\gamma} \varphi$ or $\overline{\models}^{\mathcal{M}}_{\alpha} \psi$. In the case (j), by definition 2.4 it holds that $\overline{\models}_\alpha^{\mathcal{M}} (\varphi \wedge \psi)$. In the case (jj) at every world β in $\mathcal{M}$ it holds that $\overline{\models}_\beta^{\mathcal{M}} (\varphi \wedge \psi)$ and therefore also $\overline{\models}_\alpha^{\mathcal{M}} (\varphi \wedge \psi)$.

2. $\models (\varphi \vee \psi) \leftrightarrow \neg(\neg\varphi \wedge \neg\psi)$

   (a) Suppose that for a certain world α in the model $\mathcal{M}$ it holds that $\models_\alpha^{\mathcal{M}} (\varphi \vee \psi)$. By definition 2.4 there are two cases: (j) either $\models_\alpha^{\mathcal{M}} \varphi$ or $\models_\alpha^{\mathcal{M}} \psi$, and (jj) at every world β in $\mathcal{M}$ it does not hold that $\overline{\models}_\beta^{\mathcal{M}} \varphi$ and $\overline{\models}_\alpha^{\mathcal{M}} \psi$ and there is a world γ in $\mathcal{M}$ such that it holds that either $\models_\gamma^{\mathcal{M}} \varphi$ or $\models_\gamma^{\mathcal{M}} \psi$. In the case (j) it holds that either $\overline{\models}_\alpha^{\mathcal{M}} \neg\varphi$ or $\overline{\models}_\alpha^{\mathcal{M}} \neg\psi$. In this case it holds that $\overline{\models}_\alpha^{\mathcal{M}} (\neg\varphi \wedge \neg\psi)$ and therefore $\models_\alpha^{\mathcal{M}} \neg(\neg\varphi \wedge \neg\psi)$. In the case (jj) it holds that $\overline{\models}_{\mathcal{M}} (\neg\varphi \wedge \neg\psi)$, so that $\models_{\mathcal{M}} \neg(\neg\varphi \wedge \neg\psi)$ and, in particular, $\models_\alpha^{\mathcal{M}} \neg(\neg\varphi \wedge \neg\psi)$.

   (b) Suppose that for a certain world α in the model $\mathcal{M}$ it holds that $\models_\alpha^{\mathcal{M}} \neg(\neg\varphi \wedge \neg\psi)$. By definition 2.4, there are two cases (j) either $\overline{\models}_\alpha^{\mathcal{M}} \neg\varphi$ or $\overline{\models}_\alpha^{\mathcal{M}} \neg\psi$ and (jj) at no world β in $\mathcal{M}$ it holds that at the same time $\models_\beta^{\mathcal{M}} \neg\varphi$ and $\models_\alpha^{\mathcal{M}} \neg\psi$ and there is a world γ in $\mathcal{M}$ such that either $\overline{\models}_\gamma^{\mathcal{M}} \neg\varphi$ or $\overline{\models}_\gamma^{\mathcal{M}} \neg\psi$. In the case (j) it holds that either $\models_\alpha^{\mathcal{M}} \varphi$ or $\models_\alpha^{\mathcal{M}} \psi$ and therefore it holds that $\models_\alpha^{\mathcal{M}} (\varphi \vee \psi)$. In the case (jj) $\models_{\mathcal{M}} (\varphi \vee \psi)$ and therefore $\models_\alpha^{\mathcal{M}} (\varphi \vee \psi)$.

   (c) Suppose that for a certain world α in the model $\mathcal{M}$ it holds that $\overline{\models}_\alpha^{\mathcal{M}} (\varphi \vee \psi)$. By definition 2.4, it holds that $\models_\alpha^{\mathcal{M}} \neg(\varphi \vee \psi)$ and by theorem 5.5 $\models_\alpha^{\mathcal{M}} (\neg\varphi \wedge \neg\psi)$. It follows that $\overline{\models}_\alpha^{\mathcal{M}} \neg(\neg\varphi \wedge \neg\psi)$.

   (d) Suppose that for a certain world α in the model $\mathcal{M}$ it holds that $\overline{\models}_\alpha^{\mathcal{M}} \neg(\neg\varphi \wedge \neg\psi)$. By definition 2.4 it holds that $\models_\alpha^{\mathcal{M}} (\neg\varphi \wedge \neg\psi)$ and therefore $\models_\alpha^{\mathcal{M}} \uparrow(\neg\varphi \wedge \neg\psi)$. By theorem 5.5 it holds that $\models_\alpha^{\mathcal{M}}$ $(\uparrow\neg\varphi \wedge \uparrow\neg\psi$. It follows from definition 2.4 that $\overline{\models}_\alpha^{\mathcal{M}} \uparrow(\varphi \vee \psi)$ and also $\overline{\models}_\alpha^{\mathcal{M}} (\varphi \vee \psi)$.

3. $\models \neg(\varphi \wedge \psi) \leftrightarrow (\neg\varphi \vee \neg\psi)$

   (a) Suppose that for a certain world α in the model $\mathcal{M}$ it holds that $\models_\alpha^{\mathcal{M}} \neg(\varphi \wedge \psi)$. Then, by definition 2.4, it holds that $\overline{\models}_\alpha^{\mathcal{M}} (\varphi \wedge \psi)$. By point (1) it holds that $\overline{\models}_\alpha^{\mathcal{M}} \neg(\neg\varphi \vee \neg\psi)$ and therefore $\models_\alpha^{\mathcal{M}} (\neg\varphi \vee \neg\psi)$.

   (b) Suppose that for a certain world α in the model $\mathcal{M}$ it holds that $\models_\alpha^{\mathcal{M}} (\neg\varphi \vee \neg\psi)$. By definition 2.4, it holds that $\overline{\models}_\alpha^{\mathcal{M}} \neg(\neg\varphi \vee \neg\psi)$ By what proved under (1) it holds that $\overline{\models}_\alpha^{\mathcal{M}} (\varphi \wedge \psi)$, and therefore $\models_\alpha^{\mathcal{M}} \neg(\varphi \wedge \psi)$.

(c) Suppose that for a certain world α in the model $\mathcal{M}$ it holds that $\overline{\models}^{\mathcal{M}}_{\alpha} \neg(\varphi \wedge \psi)$. By definition 2.4, it holds that $\models^{\mathcal{M}}_{\alpha} (\varphi \wedge \psi)$. By what has been proved under (1) it holds that $\models^{\mathcal{M}}_{\alpha} \neg(\neg\varphi \vee \neg\psi)$, and therefore $\overline{\models}^{\mathcal{M}}_{\alpha} \neg\varphi \vee \neg\psi)$.

(d) Suppose that for a certain world α in the model $\mathcal{M}$ it holds that $\overline{\models}^{\mathcal{M}}_{\alpha} (\neg\varphi \vee \neg\psi)$. By definition 2.4, it holds that $\models^{\mathcal{M}}_{\alpha} \neg(\neg\varphi \vee \neg\psi)$. By what has been proved under (1) it holds that $\models^{\mathcal{M}}_{\alpha} (\varphi \wedge \psi$. It follows that $\overline{\models}^{\mathcal{M}}_{\alpha} \neg(\varphi \wedge \psi)$.

4. $\models \neg(\varphi \vee \psi) \leftrightarrow (\neg\varphi \wedge \neg\psi)$

(a) Suppose that for a certain world α in the model $\mathcal{M}$ it holds that $\models^{\mathcal{M}}_{\alpha} \neg(\varphi \vee \psi)$. By definition 2.4, it holds that $\overline{\models}^{\mathcal{M}}_{\alpha} (\varphi \vee \psi)$. By what has been proved under (2) it holds that $\overline{\models}^{\mathcal{M}}_{\alpha} \neg(\neg\varphi \wedge \neg\psi)$ and therefore that $\models^{\mathcal{M}}_{\alpha} (\neg\varphi \wedge \neg\psi)$.

(b) Suppose that for a certain world α in the model $\mathcal{M}$ it holds that $\models^{\mathcal{M}}_{\alpha} (\neg\varphi \wedge \neg\psi)$. By definition 2.4, it holds that $\overline{\models}^{\mathcal{M}}_{\alpha} \neg(\neg\varphi \wedge \neg\psi)$. By what has been proved under (2) it holds that $\overline{\models}^{\mathcal{M}}_{\alpha} (\varphi \vee \psi)$ and therefore that $\models^{\mathcal{M}}_{\alpha} \neg(\varphi \vee \psi)$.

(c) Suppose that for a certain world α in the model $\mathcal{M}$ it holds that $\overline{\models}^{\mathcal{M}}_{\alpha} \neg(\varphi \vee \psi)$. By definition 2.4, it holds that $\models^{\mathcal{M}}_{\alpha} (\varphi \vee \psi)$. By what has been proved under (2) it holds that $\models^{\mathcal{M}}_{\alpha} \neg(\neg\varphi \wedge \neg\psi)$ and there fore that $\overline{\models}^{\mathcal{M}}_{\alpha} (\neg\varphi \wedge \neg\psi)$.

(d) Suppose that for a certain world α in the model $\mathcal{M}$ it holds that $\overline{\models}^{\mathcal{M}}_{\alpha} (\neg\varphi \wedge \neg\psi)$. By definition 2.4, it holds that $\models^{\mathcal{M}}_{\alpha} \neg(\neg\varphi \wedge \neg\psi)$. By what has been proved under (2) it holds that $\models^{\mathcal{M}}_{\alpha} (\varphi \vee \psi)$ and there fore that $\overline{\models}^{\mathcal{M}}_{\alpha} \neg(\varphi \vee \psi)$.

q.e.d.

Theorem 5.7

For every sentence of $\mathscr{L}$ of the form exhibited by one of the following schemas holds what follows:

T. $\models (\Box\varphi \rightarrow \varphi)$

K. $\models (\Box(\varphi \rightarrow \psi) \rightarrow (\Box\varphi \rightarrow \Box\psi))$

Df◇. $\models (\Diamond\varphi \rightarrow \neg\Box\neg\varphi)$

5. $\models (\Diamond\varphi \rightarrow \Box\Diamond\varphi)$

Proof

T. Let α a world in a model $\mathcal{M}$. We prove that $\models^{\mathcal{M}}_{\alpha} \Box\varphi \rightarrow \varphi$. According to definition 2.4 this holds if at least one of the following conditions are satisfied: (i) $\overline{\models}^{\mathcal{M}}_{\alpha}\ \Box\varphi$, (ii) $\models^{\mathcal{M}}_{\alpha}\ \varphi$, (iii) not $\models^{\mathcal{M}}_{\alpha}\ \Box\varphi$ and not $\overline{\models}^{\mathcal{M}}_{\alpha}\ \varphi$. Suppose that all conditions (i)-(iii) are not satisfied. In this case φ cannot be singular, because in such a case condition (iii) would be satisfied. If condition (i) is not satisfied it holds that $\models^{\mathcal{M}}_{\alpha}\ \Box\varphi$, so that, by theorem 2.1 for every world β in $\mathcal{M}$ it holds that $\models^{\mathcal{M}}_{\beta}\ \varphi$, so that $\models^{\mathcal{M}}_{\alpha}\ \varphi$ condition (ii) is satisfied. If condition (ii) is not satisfied then $\overline{\models}^{\mathcal{M}}_{\alpha}\ \Box\varphi$, so that condition (i) is satisfied. If condition (iii) is not satisfied either $\models^{\mathcal{M}}_{\alpha}\ \Box\varphi$ or $\overline{\models}^{\mathcal{M}}_{\alpha}\ \varphi$. However if $\models^{\mathcal{M}}_{\alpha}\ \Box\varphi$ it holds that $\models^{\mathcal{M}}_{\alpha}\ \varphi$, so that if $\overline{\models}^{\mathcal{M}}_{\alpha}\ \varphi$ it holds that $\overline{\models}^{\mathcal{M}}_{\alpha}\ \Box\varphi$ and condition (i) is satisfied. We conclude that at leat one of the conditions (i)-(iii) is satisfied.

K. Suppose that K. is not valid. Then there exists a world α in a model $\mathcal{M}$ such that not $\models^{\mathcal{M}}_{\alpha}\ (\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi))$. Since $\to$ is a two-valued truth-function by definition 2.4, in such a case it holds that $\overline{\models}^{\mathcal{M}}_{\alpha}$ $(\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi))$. By definition 2.4 at least one of the two following conditions are satisfied (i) $\models^{\mathcal{M}}_{\alpha}\ \Box(\varphi \to \psi)$ and not $\models^{\mathcal{M}}_{\alpha}\ (\Box\varphi \to \Box\psi)$, (ii) not $\overline{\models}^{\mathcal{M}}_{\alpha}\ \Box(\varphi \to \psi)$ and $\overline{\models}^{\mathcal{M}}_{\alpha}\ (\Box\varphi \to \Box\psi)$.

Suppose that condition (i) is satisfied. Then (j) for every β in $\mathcal{M}$ it holds that not $\overline{\models}^{\mathcal{M}}_{\beta}\ (\varphi \to \psi)$ which is equivalent to say that then for every β in $\mathcal{M}$ it holds that $\models^{\mathcal{M}}_{\beta}\ (\varphi \to \psi)$, and (jj) $\overline{\models}^{\mathcal{M}}_{\alpha}\ (\Box\varphi \to \Box\psi)$. Consider condition (j). In such a case it holds either that $\overline{\models}^{\mathcal{M}}_{\beta}\ \varphi$ or $\models^{\mathcal{M}}_{\beta}\ \psi$ or not $\models^{\mathcal{M}}_{\beta}\ \varphi$ and not $\overline{\models}^{\mathcal{M}}_{\beta}\ \psi$. Moreover, under condition (i), it holds that $\overline{\models}^{\mathcal{M}}_{\alpha}\ (\Box\varphi \to \Box\psi)$ which, by theorem 2.1, entails that at least one of the following conditions are satisfied: (k) $\models^{\mathcal{M}}_{\alpha}\ \Box\varphi$ and not $\models^{\mathcal{M}}_{\alpha}\ \Box\psi$, (kk) not $\overline{\models}^{\mathcal{M}}_{\alpha}\ \Box\varphi$ and $\overline{\models}^{\mathcal{M}}_{\alpha}\ \Box\psi$. Now if $\overline{\models}^{\mathcal{M}}_{\beta}\ \varphi$ for every world β, then it holds that $\overline{\models}^{\mathcal{M}}_{\alpha}\ \Box\varphi$ and rules out both $\models^{\mathcal{M}}_{\alpha}\ \Box\varphi$ and not $\overline{\models}^{\mathcal{M}}_{\alpha}\ \Box\varphi$, so that neither condition (k) nor condition (kk) may be satisfied. Suppose that $\models^{\mathcal{M}}_{\beta}\ \psi$ for every β. This rules out both not $\models^{\mathcal{M}}_{\alpha}\ \Box\psi$ and $\overline{\models}^{\mathcal{M}}_{\alpha}\ \Box\psi$. Suppose that for every world β it holds that $\models^{\mathcal{M}}_{\beta}\ \psi$. This rules out both not $\models^{\mathcal{M}}_{\alpha}\ \Box\psi$ and $\overline{\models}^{\mathcal{M}}_{\alpha}\ \Box\psi$. Suppose now that for every world β it holds that not $\models^{\mathcal{M}}_{\beta}\ \varphi$ and not $\overline{\models}^{\mathcal{M}}_{\beta}\ \psi$. This condition rules out $\models^{\mathcal{M}}_{\alpha}\ \Box\varphi$ and also with $\overline{\models}^{\mathcal{M}}_{\alpha}\ \Box\psi$. So condition (i) cannot be satisfied.

Suppose that condition (ii) is satisfied, so that it holds that not $\overline{\models}^{\mathcal{M}}_{\alpha}\ \Box(\varphi \to \psi)$ and $\overline{\models}^{\mathcal{M}}_{\alpha}\ (\Box\varphi \to \Box\psi)$. Condition not $\overline{\models}^{\mathcal{M}}_{\alpha}\ \Box(\varphi \to \psi)$ amounts to say that for no world β in $\mathcal{M}$ it holds that $\overline{\models}^{\mathcal{M}}_{\beta}\ (\varphi \to \psi)$.

Condition $\overline{\models}^{\mathcal{M}}_{\alpha}$ $(\Box\varphi \to \Box\psi)$ requires that at least one of the following conditions are satisfied: (l) $\models^{\mathcal{M}}_{\alpha} \Box\varphi$ and not $\models^{\mathcal{M}}_{\alpha} \Box\psi$, (ll) not $\overline{\models}^{\mathcal{M}}_{\alpha} \Box\varphi$ and $\overline{\models}^{\mathcal{M}}_{\alpha} \Box\psi$. Suppose that not $\overline{\models}^{\mathcal{M}}_{\alpha} \Box(\varphi \to \psi)$, so that for every world β in $\mathcal{M}$ it holds that not $\overline{\models}^{\mathcal{M}}_{\beta}$ $(\varphi \to \psi)$, which entails (being $\to$ a binary connective) that $\models^{\mathcal{M}}_{\beta}$ $(\varphi \to \psi)$ for every β. By definition 2.4 at least one of the following conditions are then satisfied for every β: (m) $\overline{\models}^{\mathcal{M}}_{\beta}$ φ, (mm) $\models^{\mathcal{M}}_{\beta}$ ψ, (mmm) not $\models^{\mathcal{M}}_{\beta}$ φ and not $\overline{\models}^{\mathcal{M}}_{\beta}$ ψ. Case (m) rules out $\models^{\mathcal{M}}_{\alpha} \Box\varphi$ and therefore also condition (l). Case (m) rules out also that not $\overline{\models}^{\mathcal{M}}_{\alpha} \Box\varphi$ and therefore rules out also condition (ll). Case (mm) rules out that not $\models^{\mathcal{M}}_{\alpha} \Box\psi$ and therefore also condition (l). Case (mm) also rules out that $\overline{\models}^{\mathcal{M}}_{\alpha} \Box\psi$ and therefore rules out also condition (ll). Case (mmm) rules out $\models^{\mathcal{M}}_{\alpha} \Box\varphi$ and therefore rule out also condition (l). Condition (mmm) also rules out that $\overline{\models}^{\mathcal{M}}_{\alpha} \Box\psi$ and therefore rules out condition (ll) as well. This concludes the proof of K.

Df$\Diamond$. By theorem 2.1 and theorem 5.2 $\models (\neg\Box\neg\varphi \leftrightarrow \neg\neg\Diamond\neg\neg\varphi)$. Applying theorem 5.2 two times and replacement equivalents with equivalents to the preceding formula we get $\models (\neg\Box\neg\varphi \leftrightarrow \Diamond\varphi)$.

5. Let α be any world in a model $\mathcal{M}$. We prove that $\models^{\mathcal{M}}_{\alpha} (\Diamond\varphi \to \Box\Diamond\varphi)$. According to definition 2.4, this holds if at least one of the following conditions are satisfied: (i) $\overline{\models}^{\mathcal{M}}_{\alpha}$ $\Diamond\varphi$, (ii) $\models^{\mathcal{M}}_{\alpha} \Box\Diamond\varphi$, (iii) not $\models^{\mathcal{M}}_{\alpha} \Diamond\varphi$ and not $\overline{\models}^{\mathcal{M}}_{\alpha}$ $\Box\Diamond\varphi$. Suppose that no condition (i)-(iii) is satisfied, so that their respective negations are *all* satisfied: (j) not $\overline{\models}^{\mathcal{M}}_{\alpha}$ $\Diamond\varphi$, (jj) not $\models^{\mathcal{M}}_{\alpha} \Box\Diamond\varphi$, (jjj) either $\models^{\mathcal{M}}_{\alpha} \Diamond\varphi$ or $\overline{\models}^{\mathcal{M}}_{\alpha}$ $\Box\Diamond\varphi$. Condition (j) may be reformulated by saying that either (j1) there exists a world β in $\mathcal{M}$ such that $\models^{\mathcal{M}}_{\beta}$ φ or (j2) no world γ in $\mathcal{M}$ is such that $\overline{\models}^{\mathcal{M}}_{\gamma}$ φ. Condition (jj) may be reformulated by saying that either (jj1) there exists a world β in $\mathcal{M}$ such that $\overline{\models}^{\mathcal{M}}_{\beta}$ $\Diamond\varphi$ or (jj2) no world γ in $\mathcal{M}$ is such that $\models^{\mathcal{M}}_{\gamma} \Diamond\varphi$. Condition (jjj) may be reformulate by saying that either (jjj1) there exists a world β in $\mathcal{M}$ such that $\models^{\mathcal{M}}_{\beta}$ φ or there exists a world γ in $\mathcal{M}$ such that $\overline{\models}^{\mathcal{M}}_{\beta}$ $\Diamond\varphi$. Suppose that (j1) is satisfied. Clearly (j1) is incompatible with condition (jj1) which entails that for no β in $\mathcal{M}$ it holds that $\models^{\mathcal{M}}_{\beta}$ φ. So necessarily, if (j1) is satisfied, (jj2) must be satisfied too, so that no γ in $\mathcal{M}$ is such that $\models^{\mathcal{M}}_{\gamma} \Diamond\varphi$. But (jj2) may be reformulate by saying that no γ in M is such that $\models^{\mathcal{M}}_{\gamma}$ φ, which again is at odds with (j1). So, let us consider (j2). Clearly, (j2) is incompatible with (jj1) which entails that there exists β in $\mathcal{M}$ such that $\overline{\models}^{\mathcal{M}}_{\beta}$ φ. So, if (j2) is satisfied, it must

be satisfied also (jj2). Now, (jj2) may be satisfied only if $\models_{\mathcal{M}} \varphi \leftrightarrow \natural$. However, this is incompatible with (jjj), which entails that there exists β in $\mathcal{M}$ such that $\models^{\mathcal{M}}_{\beta} \varphi$, while for every β in $\mathcal{M}$ it holds that not $\models^{\mathcal{M}}_{\beta} \natural$. So conditions (j), (jj) and (jjj) cannot be simultaneously satisfied. This concludes the proof of 5.

q.e.d.

Theorem 5.7 proves that our semantics encompasses S5 logic. In fact, tautologous schemas are valid on the set of ordinary sentences (which is closed with respect to the connectives $\Diamond, \Box, \wedge, \vee, \neg$), so that valid ordinary sentences may be regarded as S5-valid sentences.

Theorem 5.8

For every sentence φ of $\mathscr{L}$ it holds that $\models \varphi$ iff $\models \Box{\uparrow}\varphi$.

Proof

Suppose $\models \varphi$. Then by definition 2.13 φ is true at every world in all models. By definition 2.4 also ${\uparrow}\varphi$ is true at every world in all models. By theorem 2.1, also $\Box{\uparrow}\varphi$ is true at every world in all models, so that $\models \Box{\uparrow}\varphi$.

Suppose that $\models \Box{\uparrow}\varphi$. By Theorem 2.1 ${\uparrow}\varphi$ is true at every world in all models and by definition 2.13, φ is true at every world in all models, so that $\models \varphi$.

q.e.d.

Definition 5.3 (Quasi-atomic sentences)

A sentence φ is said to be quasi-atomic iff it is either of the form ${\uparrow}\varphi$ or of the form ${\uparrow}\neg\varphi$, where φ is either an atomic wff or $\varphi \approx \bot$.

Definition 5.4 (Normal quasi-classical form)

A sentence φ is said to be in normal quasi-classical (nqc) form iff the following conditions are satisfied:

(i) every occurrence of a subformula of the form ${\uparrow}\psi$, if any, is such that ψ is either an atomic sentence or the negation of an atomic sentence or a quasi-atomic sentence;

(ii) φ does not contain any occurrence of the conditioning symbol '|';

(iii) every occurrence of atomic sentences is immediately preceded either by the symbol '↑' or by the sequence of symbols '↑¬'.

Theorem 5.9

Every sentence in normal quasi-classical form is an ordinary sentence.

Proof

Omitted.

q.e.d.

Theorem 5.10

For every sentence φ both ${\uparrow}\varphi$ and ${\uparrow}\neg\varphi$ are logically equivalent to a sentence ψ in normal quasi-classical form.

Proof

By induction on the construction of φ. Base. If $\varphi \approx \perp$, then both $\uparrow\varphi$ and $\uparrow\neg\varphi$ are, by definition 5.3 quasi-atomic sentences, so that φ is, by definition 5.4 in nqc form. If φ is an atomic sentence then both $\uparrow\varphi$ and $\uparrow\neg\varphi$ are, by definition 5.4 in nqc form.

Step.

(a) $\varphi \approx \neg\psi$, where $\uparrow\psi$ and $\uparrow\neg\psi$ are logically equivalent, respectively, to sentences χ and ζ in nqc form.

   (i) $\uparrow\varphi$. $\uparrow\varphi \approx \uparrow\neg\psi$. But $\uparrow\neg\psi$ is assumed to be logically equivalent to ζ which is, by inductive hypothesis, in nqc form.

   (ii) $\uparrow\neg\psi$. $\uparrow\varphi \approx \uparrow\neg\neg\psi$, where $\uparrow\neg\neg\psi$ is logically equivalent, by theorem 5.2 to $\uparrow\psi$ which, in turn, is logically equivalent to χ, which is, by inductive hypothesis, in nqc form.

(b) $\varphi \approx \uparrow\psi$, where $\uparrow\psi$ and $\uparrow\neg\psi$ are logically equivalent, respectively, to sentences χ and ζ in nqc form.

   (i) $\uparrow\varphi \approx \uparrow\uparrow\psi$ which is logically equivalent to $\uparrow\psi$. Now, $\uparrow\psi$ is assumed to be logically equivalent to χ which is assumed to be, by inductive hypothesis, in nqc form.

   (ii) $\uparrow\neg\varphi \approx \uparrow\neg\uparrow\psi$ and since $\uparrow\psi$ is logically equivalent to a sentence χ in nqc form, by (a) $\neg\chi$ is also in nqc form, so that $\uparrow\neg\chi$ too, as proved above, is in nqc form.

(c) $\varphi \approx (\psi \wedge \chi)$ where $\uparrow\psi$, $\uparrow\chi$, $\uparrow\neg\psi$, and $\uparrow\neg\chi$ are equivalent, respectively, to sentences ψ_1, χ_1, ψ_2, and χ_2 in nqc form.

   (i) $\uparrow\varphi \approx \uparrow(\psi \wedge \chi)$, which, by theorem 5.5 is logically equivalent to $(\uparrow\varphi \wedge \uparrow\psi)$ that, in turn, is logically equivalent to $(\psi_1 \wedge \chi_1)$, which is, by inductive hypothesis, in nqc form.

   (ii) $\uparrow\neg(\psi \wedge \chi)$ is, by theorem 5.6, logically equivalent to $\uparrow(\neg\psi \vee \neg\chi)$, which is, in turn, logically equivalent to $((\uparrow\psi \vee \uparrow\chi) \vee (\neg\Diamond(\uparrow\neg\psi \wedge \uparrow\neg\chi) \wedge (\Diamond\uparrow\psi \vee \Diamond\uparrow\chi)))$ which is logically equivalent to $((\psi_1 \vee \chi_1) \vee (\neg\Diamond(\psi_2 \wedge \chi_2) \wedge (\Diamond\psi_1 \vee \Diamond\chi_1)))$ which is, in the presence of the inductive hypothesis, logically equivalent to a sentence in nqc form.

(d) $\varphi \approx \Diamond\psi$, where $\uparrow\psi$ is logically equivalent to a sentence χ in nqc form.

   (i) By theorem 5.5, $\uparrow\varphi \approx \uparrow\Diamond\psi$ is logically equivalent to $\Diamond\uparrow\psi \approx \Diamond\chi$. Now, if χ is in nqc form also $\Diamond\chi$ and $\Diamond\uparrow\psi \approx \Diamond\chi$ are in nqc form in the presence of the inductive hypothesis.

   (ii) $\uparrow\neg\varphi \approx \uparrow\neg\Diamond\psi$, which is logically equivalent, by theorem 5.5, to $(\uparrow(\psi \vee \neg\psi) \wedge \neg\Diamond\uparrow\psi)$. Now, $\uparrow(\psi \vee \neg\psi)$ is logically equivalent, by theorem 5.5 to $((\uparrow\psi \vee \uparrow\neg\psi) \vee (\neg\Diamond\uparrow\neg\psi \wedge \Diamond\uparrow\psi))$, which, in virtue of the inductive hypothesis, is logical equivalent to a nqc sentence.

(e) $\varphi \approx (\psi \vee \chi)$, where $\uparrow\psi$ and $\uparrow\chi$, are logically equivalent, respectively, to sentences ψ_1, χ_1, ψ_2, and χ_2 in nqc form.

  (i) By theorem 5.5, $\uparrow(\psi \vee \chi)$ is logically equivalent to $((\uparrow\psi \vee \chi_1) \vee (\neg\Diamond(\psi_2 \wedge \chi_2) \wedge (\Diamond\psi_1 \vee \Diamond\chi_1)))$ which is, by the inductive hypothesis, logically equivalent to a sentence in nqc form.

  (ii) By theorem 5.6, $\uparrow\neg(\psi \vee \chi)$ is, logically equivalent to $\uparrow(\neg\psi \wedge \neg\chi)$. Now, $\uparrow(\neg\psi\wedge\neg\chi)$ is logically equivalent, by theorem 5.5 to $(\uparrow\neg\psi\wedge \uparrow\neg\chi)$ which is, in turn, logically equivalent to $(\psi_2 \wedge \chi_2)$, which is, by the inductive hypothesis, in nqc form.

(f) $\varphi \approx (\psi \rightarrow \chi)$ where $\uparrow\psi$, $\uparrow\chi$, $\uparrow\neg\psi$, $\uparrow\neg\chi$ are logically equivalent, respectively, to ψ_1, χ_1, ψ_2, χ_2 in nqc form.

  (i) By theorem 5.5, $\uparrow(\psi \rightarrow \chi)$ is logically equivalent to $((\uparrow\chi \vee (\neg\uparrow\psi \wedge \neg\uparrow\neg\chi)) \vee \uparrow\neg\psi)$ which is logically equivalent to $(\chi_1 \vee (\neg\psi_1 \wedge \neg\chi_2) \vee \psi_2)$ which is, by the inductive hypothesis, in nqc form.

  (ii) $\uparrow\neg(\psi \rightarrow \chi)$ is logically equivalent to $((\psi_1 \wedge \neg\chi_1) \vee (\chi_2 \wedge \neg\psi_2))$ which is, by inductive hypothesis, in nqc form.

(g) $\varphi \approx (\chi \mid \psi)$, where $\uparrow\psi$, $\uparrow\chi$, $\uparrow\neg\psi$, $\uparrow\neg\chi$ are logically equivalent, respectively, to ψ_1, χ_1, ψ_2, χ_2 in nqc form.

  (i) By theorem 5.5, $\uparrow\varphi$ is logically equivalent to $((\uparrow\psi\wedge\uparrow\chi)\vee(\Diamond(\uparrow\psi\wedge \uparrow\chi) \wedge \neg\Diamond(\uparrow\psi \wedge \uparrow\neg\chi)))$ which, in turn, is logically equivalent to $((\psi_1 \wedge \chi_1) \vee (\Diamond(\psi_1 \wedge \chi_1) \wedge \neg\Diamond(\psi_1 \wedge \chi_2))$, which is, by inductive hypothesis, in nqc form.

  (ii) $\uparrow\neg\varphi$ is logically equivalent to $\uparrow(\chi_2 \mid \psi)$, and it follows from what has been proved above that it is logically equivalent to a sentence in nqc form.

q.e.d.

At this point we will introduce two sub-languages, named respectively $\mathscr{L}_1$ and $\mathscr{L}_2$. The language $\mathscr{L}_1$ has all the symbols of $\mathscr{L}$ except the conditioning symbol '|'. Moreover, the atomic sentences of $\mathscr{L}_1$ are the quasi-atomic sentences of $\mathscr{L}$. All the sentences of $\mathscr{L}_1$ are ordinary sentences and, from a semantical point of view, essentially two-valued (that is two-valued in every model). The language $\mathscr{L}_2$ is just a classic modal sentential language. It lacks, of course, the symbol '|'.

Definition 5.5 (The sub-language $\mathscr{L}_1$)

Let QA the set of all the quasi-atomic sentences of $\mathscr{L}$. The class of sentences of $\mathscr{L}_1$ is recursively defined as follows:

1. $\bot$ is a sentence of $\mathscr{L}_1$.
2. Every quasi-atomic sentence of $\mathscr{L}$ is a sentence of $\mathscr{L}_1$.
3. If φ is a sentence of $\mathscr{L}_1$ then also $\neg\varphi$ and $\Diamond\varphi$ are sentences of $\mathscr{L}_1$.

4. If φ and ψ are sentences of $\mathscr{L}_1$ then also $(\varphi \vee \psi)$, $(\varphi \wedge \psi)$, $(\varphi \rightarrow \psi)$ are sentences of $\mathscr{L}_1$.

5. Nothing else is a sentence of $\mathscr{L}_1$.

Theorem 5.11

Every sentence in normal quasi-classical form is a sentence of $\mathscr{L}_1$.

Proof
Omitted.

q.e.d.

Theorem 5.12

Every sentence in $\mathscr{L}_1$ is an ordinary sentence.

Proof
Omitted.

q.e.d.

Definition 5.6 (The sub-language $\mathscr{L}_2$)

The class of sentences of $\mathscr{L}_2$ is recursively defined as follows:

1. $\perp$ is a sentence of $\mathscr{L}_2$.

2. Every atomic sentence of $\mathscr{L}$ is a sentence of $\mathscr{L}_2$.

3. If φ is a sentence of $\mathscr{L}_2$ then also $\neg\varphi$ and $\Diamond\varphi$ are sentences of $\mathscr{L}_2$.

4. If φ and ψ are sentences of $\mathscr{L}_2$ then also $(\varphi \vee \psi)$, $(\varphi \wedge \psi)$, $(\varphi \rightarrow \psi)$ are sentences of $\mathscr{L}_2$

5. Nothing else is a sentence of $\mathscr{L}_2$

Theorem 5.13

Let $\mathfrak{M}$ the class of models of $\mathscr{L}$ such that for every atomic sentence $\mathbb{P}_n$ it holds that $P_n = W$. Then a sentence φ of $\mathscr{L}_2$ is valid in every model in $\mathfrak{M}$ iff it is a theorem of the modal system S5.

Proof
Omitted.

q.e.d.

Theorem 5.14

Every sentence φ of $\mathscr{L}$ is logically equivalent to a syntactically simple tri-event.

Proof
By Theorem 5.5, φ is logically equivalent to $(\uparrow\varphi \mid \updownarrow\varphi)$. By Theorem 5.10 we can assume, without loss of generality, that $\uparrow\varphi$ and $\updownarrow\varphi$ are in nqc form. From this follows, by definition 5.1 that $(\uparrow\varphi \mid \updownarrow\varphi)$ is a syntactically simple tri-event.

q.e.d.

Theorem 5.15

For every sentence φ of $\mathscr{L}$, if $\models \varphi$ then there is a sentence ψ of $\mathscr{L}_2$, effectively associated to φ, which, by definition 5.6, is valid in those models of $\mathscr{L}$ in which for every atomic sentence $\mathbb{P}_n$ occurring in φ it holds that $P_n = W$.

Proof

Suppose $\models \varphi$. By Theorem 5.8 we may assume, without loss of generality that $\varphi = \uparrow\Box\varphi$. By Theorem 5.10 φ is logically equivalent to a sentence χ in nqc form. Now let $\mathbb{P}_1 \ldots \mathbb{P}_k$ the distinct atomic sentences occurring in χ. Let us associate with every $\mathbb{P}_i (1 \leq i \leq k)$ two atomic sentences $\mathbb{P}_i^1$ and $\mathbb{P}_i^2$ not belonging to the set $\{\mathbb{P}_1, \ldots, \mathbb{P}_k\}$. Let ψ' be obtained by replacing, for every $i (1 \leq i \leq k)$ in χ every occurrence of $\uparrow\mathbb{P}_i\}$ by $\mathbb{P}_i^1$ and every occurrence of $\uparrow\mathbb{P}_i$ by $\mathbb{P}_i^2$. Let ψ'' be the sentence $(\neg\Diamond(\mathbb{P}_1^1 \wedge \mathbb{P}_1^2) \wedge \ldots \wedge \neg\Diamond(\mathbb{P}_k^1 \wedge \mathbb{P}_k^2))$. Let ψ be the sentence $(\psi'' \rightarrow \psi')$. ψ is a sentence of $\mathscr{L}_2$. We prove that $\models_{\mathcal{M}} \psi$ for every model $\mathcal{M}$ such that for every atomic sentence $\mathbb{P}_n$ occurring in φ it holds that $P_n = W$. Under the hypothesis that $\models \varphi$, it holds that $\models_{\mathcal{M}} \varphi$. Given two models $\mathcal{M}_1$ and $\mathcal{M}_2$, let us consider the following relation $\mathcal{M}_1 \; R_\varphi \; \mathcal{M}_2$ between $\mathcal{M}_1$ and $\mathcal{M}_2$ that holds iff (1) $\mathcal{M}_1$ and $\mathcal{M}_2$ share the same set of worlds W, (2) for every world $\alpha \in W$, and every quasi-atomic sub-sentence ζ of $\models_\alpha^{\mathcal{M}_1} \zeta$ iff $\models_\alpha^{\mathcal{M}_2} \zeta$, (3) for every world $\alpha \in W$ and every quasi-atomic sub-sentence ζ of φ, $\overline{\models}_\alpha^{\mathcal{M}_1} \psi$ iff $\overline{\models}_\alpha^{\mathcal{M}_2} \psi$. R_φ is an equivalence relation, according to which the set of models are partitioned in equivalences classes. Now, every quasi-atomic sentence of the form $\uparrow\mathbb{P}_i$ occurring in φ is true at a given world α (in any model $\mathcal{M}$) iff the quasi-atomic sentence $\uparrow\neg\mathbb{P}_i$ is false at α, and it is false at α iff $\uparrow\neg\mathbb{P}_i$ is true at α. So there is a one-to-one correspondence between the set of the equivalence classes and the set of those models $\mathcal{M}$ such that $P_n = W$ and for every world α in $\mathcal{M}$ it holds that $\models_\alpha^{\mathcal{M}} \mathbb{P}_i^1$ iff $\models_\alpha^{\mathcal{M}} \uparrow\mathbb{P}_i$ and $\overline{\models}_\alpha^{\mathcal{M}} \mathbb{P}_i^2$ iff $\overline{\models}_\alpha^{\mathcal{M}} \uparrow\neg\mathbb{P}_i$. We may associate to each equivalence class a single model satisfying that property. φ and ψ have the same truth-value at each world in the corresponding models. Since $\models \varphi$ also $\models_\alpha^{\mathcal{M}} \psi'$ for every world α in such models. Let $\mathcal{M}'$ be a residual model in which for some world and some i, both $\mathbb{P}_i^1$ and $\mathbb{P}_i^2$ are true. In such a case ψ'' is false and therefore ψ is true. So ψ is true at every world in every model such that $P_n = W$.

q.e.d.

Corollary 5.1

For every sentence φ of $\mathscr{L}$, if $\models \varphi$ then there is a sentence ψ of $\mathscr{L}_2$, effectively associated to φ, which is a theorem of the modal system S5.

Proof

Immediate from theorem 5.15, theorem 5.13 and the completeness theorem for S5.

q.e.d.

Corollary 5.2

There is a decision procedure for satisfiability in every model, validity in every model and logical consequence among the sentences of $\mathscr{L}$.

Proof

Immediate from corollary 5.1, and the fact that S5 logic is decidable.

q.e.d.

6. Probability

6.1 Probability of Tri-events in General

In the original de Finetti's theory of probability, the probability is first defined for ordinary events and as a second step extended to any tri-event by the equation $\frac{\mathbf{P}(\uparrow\varphi)}{\mathbf{P}(\updownarrow\varphi)}$ provided $\mathbf{P}(\updownarrow\varphi) > 0$. The axioms of probability for two-valued sentences are the standard axioms for finite probability. This definition applies unmodified also to our modal theory, even if the truth conditions of φ (and therefore of $\uparrow\varphi$ and $\updownarrow\varphi$) are different. Quasi-tautologies (that are converted into valid sentences in all models of our theory) have probability 1 in de Finetti's theory as in our theory. Dually, quasi-contradictions (that are converted in countervalid sentences in all models in our theory) have probability 0 in de Finetti's theory as in our theory. Modal sentences — that is sentences of the form $\Diamond\varphi$ or $\Box\varphi$ — that are absent in de Finetti's theory, have either probability 1 or probability 0.

An equivalent alternative approach is to provide a set of axioms over all tri-events that is a generalisation of the standard theory. Several explanations of Lewis' triviality results assume that the probability of conditionals obeys the standard laws of probability. As already observed, this assumption is natural if we consider conditional sentences as two-valued sentences obeying the standard logic but is no longer reasonable if we represent conditionals as tri-events. The standard axioms of probability apply to Boolean algebras of sentences (up to logical equivalence), while the corresponding algebraic structure for tri-events is *not* a Boolean algebra. What we need is a more general theory of probability such that naturally comes down to the standard probability concerning ordinary sentences.

Since our theory is throughout a modal theory, and the modal notion involved in probability theory are model-relative, we propose a set of axioms regarding a given model $\mathcal{M}$. The model-relative probability function will be denoted $\mathbf{P}^{\mathcal{M}}$. Probability is defined for every sentence of $\mathscr{L}$ except for singular sentences. For every sentence φ of $\mathscr{L}$, the probability $\mathbf{P}^{\mathcal{M}}(\varphi)$ is considered *undefined*.[11] Accordingly, the values of the metalinguistic variables φ and ψ below are assumed to be such that for every α in $\mathcal{M}$ it holds that $\models^{\mathcal{M}}_{\alpha} \Diamond\updownarrow\varphi$ and $\models^{\mathcal{M}}_{\alpha} \Diamond \updownarrow \psi$, that is we assume that neither φ nor ψ are singular in $\mathcal{M}$.

A1 $\mathbf{P}^{\mathcal{M}}(\varphi) \geq 0$

A2 $\mathbf{P}^{\mathcal{M}}(\bot) = 0$

A3 $\mathbf{P}^{\mathcal{M}}(\uparrow\varphi) = \mathbf{P}^{\mathcal{M}}(\varphi) \times \mathbf{P}^{\mathcal{M}}(\updownarrow\varphi)$

A4 If ψ is an $\mathcal{M}$-consequence of φ, $\mathbf{P}^{\mathcal{M}}(\varphi) \leq \mathbf{P}^{\mathcal{M}}(\psi)$

[11] We explain in the next section why the closure move of assigning probability 1 to every singular sentence is not appropriate in the current approach.

A5 $\mathbf{P}^{\mathcal{M}}(\neg\varphi) = 1 - \mathbf{P}^{\mathcal{M}}(\varphi)$

Theorem 6.1

Let $\mathcal{M}$ be a model. Let $\varphi = (\psi \mid \chi)$ where ψ and χ express two-valued propositions in $\mathcal{M}$. Let $\mathbf{P}^{\mathcal{M}}$ a probability function over $\mathcal{M}$ such that $\mathbf{P}^{\mathcal{M}}(\chi) > 0$. Then it holds that

$$\mathbf{P}^{\mathcal{M}}(\varphi) = \frac{\mathbf{P}^{\mathcal{M}}(\psi \wedge \chi)}{\mathbf{P}^{\mathcal{M}}(\chi)}$$

Proof

By definition 2.4 $\models^{\mathcal{M}}_{\alpha} \uparrow\varphi$ iff $\models^{\mathcal{M}}_{\alpha} \psi$ and $\models^{\mathcal{M}}_{\alpha} \chi$, that is iff $\models^{\mathcal{M}}_{\alpha} \uparrow\psi$ and $\models^{\mathcal{M}}_{\alpha} \uparrow\chi$. Now, $\uparrow\psi$ and $\uparrow\chi$ are, respectively, logically $\mathcal{M}$–equivalent to ψ and χ, since they express two-valued propositions. By Theorem 5.5 $\uparrow(\psi \wedge \chi)$ is logically equivalent to $(\uparrow\psi\uparrow\chi)$. Since $(\uparrow\psi \wedge \uparrow\chi)$ is $\mathcal{M}$–equivalent to $(\psi \wedge \chi)$, $\uparrow\varphi$ is $\mathcal{M}$–equivalent to $(\psi \wedge \chi)$. Consider $\updownarrow\varphi$. By theorem 2.1 $\updownarrow\varphi$ iff either $\models^{\mathcal{M}}_{\alpha} \varphi$ or $\overline{\models}^{\mathcal{M}}_{\alpha} \varphi$. In both cases $\models^{\mathcal{M}}_{\alpha} \updownarrow\varphi$ iff $\models^{\mathcal{M}}_{\alpha} \chi$, so that $\updownarrow\varphi$ is logically equivalent to χ. By axiom A3, it holds that

$$\mathbf{P}^{\mathcal{M}}(\varphi) = \frac{\mathbf{P}^{\mathcal{M}}(\uparrow\varphi)}{\mathbf{P}^{\mathcal{M}}(\updownarrow\varphi)} = \frac{\mathbf{P}^{\mathcal{M}}(\psi \wedge \chi)}{\mathbf{P}^{\mathcal{M}}(\chi)}$$

q.e.d.

6.2 *p*–Entailment and *p*-Consistency

Adams oscillates between two notions of *p*-consistency. In his 1975 book, his definition was based on the idea that the probability of a conditional is *proper* only if the antecedent has a probability greater than 0. The definition is extended to a set X of formulas so that an assignment is proper iff it is proper to every conditional formula of X (Adams 1975: 49-50). *p*-consistency and *p*-inconsistency are defined considering only proper assignments. More exactly, X is *p*-consistent iff for all real numbers $\epsilon > 0$ there exist probability assignments p which are proper for X such that $\mathbf{P}(\varphi) \geq 1 - \epsilon$ for every element φ of X.

As a result, a set of conditionals of the form $\{\varphi \Rightarrow \psi, \varphi \Rightarrow \neg\psi\}$ is considered as *p*-inconsistent. Later, Adams changed his mind and returning to his previous view (Adams 1965, 1966), got rid of the notion of proper assignment and assumed again that when $\mathbf{P}(\varphi) = 0$, $\mathbf{P}(\psi \mid \varphi) = 1$. In such a case, the set $\{\varphi \Rightarrow \psi, \varphi \Rightarrow \neg\psi\}$ is considered as *p*-consistent. However, Adams remarked that

> [P]robabilistic consistency is considerably more difficult to characterize, especially because it has more than one sense. In one sense A⇒B and A⇒ ∼B are consistent, since they are both certain when p(A) = 0, and therefore p(A⇒B) = p(A⇒ ∼B) = 1. However, there seems to be another sense, for instance in which ‘If Anne goes to the party then Ben will go, and if she goes to the party then Ben won’t go’ would be

'absurd' even though p(A⇒B) and p(A⇒ ∼B) can both equal 1, and it is important to make this sense clear (Adams 1998: 181).

McGee (1994) adopted the so-called Popper-functions to allow conditional probability $\mathbf{P}(\psi \mid \varphi)$ to be defined even when φ is impossible. According to Popper functions $\mathbf{P}(\varphi \mid \bot) = 1$ for every sentence φ, but may have any value if φ is a consistent sentence such that $\mathbf{P}(\varphi) = 0$. In our context, we *must* drop the assumption that $\mathbf{P}(\varphi \mid \bot) = 1$, because it would imply that $\mathbf{P}(\natural) = \mathbf{P}(\top)$, so that two logically nonequivalent sentences would have the same probability value for every probability function.[12] Moreover, if the probability of an indicative conditional is the conditional expectation of a truth-value, clearly it is undefined when the antecedent is impossibly true. One is tempted to considered undefined the probability for every tri-event φ that, in a given model $\mathcal{M}$, is such that $\mathbf{P}(\updownarrow\varphi) = 0$. Indeed, if there is a reasonable certainty that a tri-event has no truth value, it is natural to assume that it has no probability value too. $\mathbf{P}(\updownarrow\varphi) = 0$ is always true when in a conditional sentence of the form $(\psi \mid \varphi)$, it holds that $\mathbf{P}(\varphi) = 0$. o, this assumption corresponds to the standard assumption that conditional probability is undefined when the conditioning event has probability 0. However, our axioms for probability do not entail this. So, we prefer to stick with Adams 1975 approach, defining *p*-entailment and *p*-consistency in terms of those probability assignments that assign a probability greater than 0 to every sentence of the form $\updownarrow\varphi$ for every involved sentence φ, while allowing probability functions to be defined even when the antecedent has probability 0 (short of being countervalid).

In the light of these considerations, we may define *p*-entailment and *p*-consistency for tri-events as follows.

Definition 6.1 (*p*-entailment in a model $\mathcal{M}$)

Let $X \cup \{\varphi\}$ be a set of sentences of $\mathscr{L}$. Let $\mathcal{M}$ be a model. X probabilistically entails φ in $\mathcal{M}$ iff either

(A) for all $\epsilon > 0$ there exists $\delta > 0$ such that for all coherent probability assignments $\mathbf{P}^{\mathcal{M}}$ for $\mathscr{L}$ such that for every element ψ belonging to X such that $\mathbf{P}^{\mathcal{M}}(\updownarrow \psi) > 0$, it holds that if $\mathbf{P}^{\mathcal{M}}(\chi) \geq 1 - \delta$ for all $\chi \in X$, then $\mathbf{P}^{\mathcal{M}}(\varphi) \geq 1 - \epsilon$
or

(B) $X \cup \{\varphi\}$ is singular.

Condition (b) is not present in the original Adams definition because singular sentences are not present in the language that he considered, and we have added clause (b) for closure reasons.

[12] This happens also in Adams' probability logic (1965, 1966, 1998), while does not happen in McGee's probability logic (1994).

Definition 6.2 (*p*-consistency in a model $\mathcal{M}$)

X be a finite set of sentences of $\mathscr{L}$. Let $\mathcal{M}$ be a model. X is probabilistically consistent iff for all real numbers $\epsilon > 0$, there exists at least a probability assignment such that for every element φ of X it holds that $\mathbf{P}^{\mathcal{M}}(\updownarrow \varphi) > 0$ and $\mathbf{P}^{\mathcal{M}}(\varphi) \geq 1 - \epsilon$.

6.3 Satisfiability and Adams' Confirmability

We may compare our satisfiability notion to Adams' *confirmability*. We prove the link between the two notions by the next theorem. Confirmability, in turn, may be easily defined concerning a given model. Let Γ be a finite set of syntactically simple sentences of the form $(\psi \mid \varphi)$ where φ is not countervalid. We may assume, without loss of generality, that φ and ψ are in nqc form. Let Γ' the set of the sentences in nqc form occurring in at least a sentence of Γ. Let $\mathcal{M}$ be a model. Every element of Γ' is either true or false at every world in $\mathcal{M}$. For every world α in $\mathcal{M}$, we define an assignment $t_\alpha^{\mathcal{M}}$ of truth-values to the elements of Γ' in the following way: $t_\alpha^{\mathcal{M}}(\varphi) =$ true iff $\models_\alpha^{\mathcal{M}} \varphi$ and $t_\alpha^{\mathcal{M}}(\varphi) =$ false iff $\overline{\models}_\alpha^{\mathcal{M}} \varphi$. We say that an element $(\psi \mid \varphi)$ of Γ is *verified* under the truth-assignment $t_\alpha^{\mathcal{M}}$ iff $\models_\alpha^{\mathcal{M}} \varphi$ and $\models_\alpha^{\mathcal{M}} \psi$, that $(\psi \mid \varphi)$ is *falsified* under the truth-assignment $t_\alpha^{\mathcal{M}}$ iff $\models_\alpha^{\mathcal{M}} \varphi$ and $\overline{\models}_\alpha^{\mathcal{M}} \psi$, and that $(\psi \mid \varphi)$ is neither verified nor falsified iff $\overline{\models}_\alpha^{\mathcal{M}} \varphi$. Notice that this notion of verification does not coincide with truth in the model $\mathcal{M}$. For example, if $\models_{\mathcal{M}} \psi$ and φ is false at a given world α, $t_\alpha^{\mathcal{M}}(\psi \mid \varphi)$ is neither verified nor falsified, while being true at every world, according to our semantics. We say that Γ is *confirmed* by $t_\alpha^{\mathcal{M}}$ iff, under the truth-assignment $t_\alpha^{\mathcal{M}}$, no member of Γ is falsified and at least one is verified. We say that Γ is *disconfirmed* by $t_\alpha^{\mathcal{M}}$ iff, under the truth-assignment $t_\alpha^{\mathcal{M}}$, no member of Γ is verified and at least one is falsified. We say that Γ is *confirmable* at the model $\mathcal{M}$ iff there exists a truth-assignment $t_\alpha^{\mathcal{M}}$ that confirms it and is *disconfirmable* at the model $\mathcal{M}$ iff there exists a truth-assignment that disconfirms it.

Theorem 6.2

Let Γ be a finite set of syntactically simple tri-events of the form $(\psi \mid \varphi)$ where φ and ψ are in nqc form and φ is not countervalid in $\mathcal{M}$. Γ is satisfiable at a model $\mathcal{M}$ iff every non-empty subset of Γ is confirmable at $\mathcal{M}$.

Proof

We prove first (a) that if Γ is satisfiable at a model $\mathcal{M}$, then every non-empty subset of Γ is confirmable at $\mathcal{M}$, and (b) that if every non-empty subset of Γ is confirmable at $\mathcal{M}$, then Γ is satisfiable at $\mathcal{M}$.

(a) Suppose that Γ is satisfiable at a model $\mathcal{M}$. If Γ is empty, then it is trivially confirmable. No element of Γ is singular by hypothesis. By definition 2.7, for every nonempty subset Γ' of Γ there is a world α in $\mathcal{M}$ such that for every element φ in Γ' it holds that not $\overline{\models}_\alpha^{\mathcal{M}} \varphi$ and there is an element ψ in Γ' such that $\models_\alpha^{\mathcal{M}} \psi$. Suppose $\psi = (\xi \mid \chi)$. There are two cases: (i) $\models_\alpha^{\mathcal{M}} \chi$ and $\models_\alpha^{\mathcal{M}} \xi$, and (ii) $\models_\alpha^{\mathcal{M}} \psi$ but either not $\models_\alpha^{\mathcal{M}} \xi$

or not $\models^{\mathcal{M}}_{\alpha} \chi$. In the case (i) Γ' is confirmed by the truth-assignment $t^{\mathcal{M}}_{\alpha}$, in the case (ii) ψ is a near-tautology, so that at every world α in $\mathcal{M}$ it holds either it is verified or falsified by $t^{\mathcal{M}}_{\alpha}$. Suppose *ab absurdo* that for every world β at which both $\models^{\mathcal{M}}_{\beta} \chi$ and $\models^{\mathcal{M}}_{\beta} \xi$ hold, there is an element ζ of Γ' such that it holds $\overline{\models}^{\mathcal{M}}_{\beta}$ β. If ζ is a near-contradiction, then ζ is countervalid because we assumed that it is not singular. In this case, it is false at every world, which contradicts the hypothesis that Γ is satisfiable. If ζ is not a near-tautology then there is a world γ at which it holds that $\models^{\mathcal{M}}_{\gamma} \zeta$ is true while not $\overline{\models}^{\mathcal{M}}_{\gamma}$ ψ. This result holds for every other element of Γ'. However, if so Γ' is confirmed by at least one truth-assignment, and therefore is confirmable.

(b) Suppose that every non-empty subset Γ' of Γ is confirmable at $\mathcal{M}$. This means that there exists a world α in $\mathcal{M}$ such that under the truth-assignment $t^{\mathcal{M}}_{\alpha}$, no element of Γ' is falsified and at least one of them is verified. Now every not falsified element φ is such that not $\overline{\models}^{\mathcal{M}}_{\alpha}$ φ and every verified element is such that $\models^{\mathcal{M}}_{\alpha} \varphi$. It follows that Γ is satisfiable at $\mathcal{M}$.

q.e.d.

In the light of theorem 5.14, the assumptions that the elements of Γ are simple and that φ and ψ are in nqc form implies no loss of generality. The condition that the antecedent is not countervalid in $\mathcal{M}$ may be expressed by the condition that, for each element φ of Γ, $\updownarrow\varphi$ is not countervalid in $\mathcal{M}$.

6.4 *p*–Entailment and Logical Consequence

The link between our semantics and probabilistic semantics is easy to establish by means of the Adams' notion of *yielding* that is defined in terms of the Adams' notions explained in the precedent section. It may be easily adapted to our semantics and relativized to a model $\mathcal{M}$. Let $\Gamma \cup \{\varphi\}$ be a finite set of sentences of the form $(\psi \mid \chi)$ where χ and ψ are in nqc form and χ is not countervalid in $\mathcal{M}$. Γ *yields* φ in $\mathcal{M}$ iff (i) every truth-assignment $t^{\mathcal{M}}_{\alpha}$ that confirms Γ verifies φ, and (ii) every truth-assignment $t^{\mathcal{M}}_{\alpha}$ that falsifies no element of Γ does not falsify φ. Adams has proved that Γ *p*-entails φ iff either Γ is *p*-inconsistent, or there is a subset Γ' of Γ that yields φ. Adapting his proof to the preceding model-relative notions is straightforward.

Theorem 6.3

Let $\mathcal{M}$ be a model. Let $\Gamma \cup \{\varphi\}$ be a finite set of sentences syntactically simple sentences of the form $(\psi \mid \chi)$, where χ and ψ are in nqc form and φ is not countervalid in $\mathcal{M}$. We prove first (a) that if $\Gamma \models_{\mathcal{M}} \varphi$, then either Γ is unsatisfiable in $\mathcal{M}$ or some subset Γ' of Γ yields φ in $t^{\mathcal{M}}_{\alpha}$ and (b) that if Γ is unsatisfiable in $t^{\mathcal{M}}_{\alpha}$ or some subset Γ' of Γ yields φ in $t^{\mathcal{M}}_{\alpha}$, then $\Gamma \models_{\mathcal{M}} \varphi$.

Proof

(A) Suppose that $\Gamma \models_{\mathcal{M}} \varphi$. If $\models_{\mathcal{M}} \varphi$ or Γ is unsatisfiable, then trivially Γ yields φ, so that the theorem immediately is proved in this case. If $\Gamma = \emptyset$ and not $\models_{\mathcal{M}} \varphi$ then not $\Gamma \models_{\mathcal{M}} \varphi$. Suppose that $\Gamma \neq \emptyset$ so that there is a non-empty subset Γ' of Γ such that for every world α in $\mathcal{M}$ it holds: (i) for every $\psi \in \Gamma'$ it holds that not $\overline{\models}_{\alpha}^{\mathcal{M}} \psi$ then not $\overline{\models}_{\alpha}^{\mathcal{M}} \varphi$, (ii) if for every $\psi \in \Gamma'$ it holds that not $\overline{\models}_{\alpha}^{\mathcal{M}} \psi$ and for at least one element ψ of Γ' if it holds that $\models_{\alpha}^{\mathcal{M}} \psi$ then it holds that $\models_{\alpha}^{\mathcal{M}} \varphi$. Consider the case (i). If for every $\psi \in \Gamma'$ it holds that not $\overline{\models}_{\alpha}^{\mathcal{M}} \psi$ then ψ is not falsified at α. And if it holds that not $\overline{\models}_{\alpha}^{\mathcal{M}} \varphi$, then φ is not falsified at α. Consider case (ii). Suppose that for every $\psi \in \Gamma'$ it holds that not $\overline{\models}_{\alpha}^{\mathcal{M}} \psi$ and for at least one element ψ of Γ' it holds that $\models_{\alpha}^{\mathcal{M}} \psi$. Since $\models_{\alpha}^{\mathcal{M}} \varphi$, φ is either verified at α or ψ is a near-tautology in $\mathcal{M}$. If φ is a near-tautology in $\mathcal{M}$, then it is not falsified at every world, and there is a world at which it is verified. In this case $\emptyset$ yields φ and the condition is satisfied.

(B) Suppose that either Γ is unsatisfiable or some subset of Γ yields φ in $\mathcal{M}$. If Γ is unsatisfiable then $\Gamma \models_{\mathcal{M}} \varphi$ by definition. If $\Gamma = \emptyset$, then φ is such that no truth-assignment falsifies it and at least one verifies it. In this case φ is valid in $\mathcal{M}$, so that by definition 2.15 φ is an $\mathcal{M}$-consequence of Γ. Otherwise, let Γ' be a subset of Γ yielding φ. By hypothesis, every truth-assignment that falsifies no element of Γ does not falsify φ. Since every element which is not false at some world α in $\mathcal{M}$ is not falsified by the truth-assignment $t_{\alpha}^{\mathcal{M}}$, it follows that for every $\psi \in \Gamma'$ it holds that not $\overline{\models}_{\alpha}^{\mathcal{M}} \psi$ then it holds that not $\overline{\models}_{\alpha}^{\mathcal{M}} \varphi$. Suppose now that, for every world α in $\mathcal{M}$, every truth-assignment $t_{\alpha}^{\mathcal{M}}$ confirming Γ' verifies φ. For every world α at which no elements of Γ' is falsified by the truth-assignment $t_{\alpha}^{\mathcal{M}}$ but some element ψ of Γ' is verified by $t_{\alpha}^{\mathcal{M}}$, it holds that $\models_{\alpha}^{\mathcal{M}} \psi$. Moreover, by hypothesis, also φ is verified by $t_{\alpha}^{\mathcal{M}}$. In this case it holds that $\models_{\alpha}^{\mathcal{M}} \varphi$. Suppose there is a world β at which it holds that $\models_{\beta}^{\mathcal{M}} \psi$ while every other element χ of Γ' is such that it holds that not $\overline{\models}_{\beta}^{\mathcal{M}} \chi$ and suppose that ψ is not verified by the truth-assignment $t_{\beta}^{\mathcal{M}}$. In this case ψ is $\mathcal{M}$-valid. Let Δ be the subset of Γ' of those sentences of Γ' that are $\mathcal{M}$-valid. Consider the set $\Gamma'' = \Gamma' - \Delta$. The set of worlds at which the elements of Γ'' are true coincides with the set of worlds at which the elements of Γ'' are verified. Hence $\Gamma'' \models_{\mathcal{M}} \varphi$, and since $\Gamma'' \subseteq \Gamma$ it holds that $\Gamma \models_{\mathcal{M}} \varphi$.

q.e.d.

Theorem 6.3 does not consider the case in which every element of $\Gamma \cup \{\varphi\}$ is singular. However, in such a case, according to *our* definitions it holds that $\Gamma \models_{\mathcal{M}} \varphi$ iff Γ *p*-entails φ.

7. Assertion of Conditionals

'Assertion' denotes a pragmatic notion. Asserting a two-valued sentence φ (or rather, the proposition it expresses) is *categorically* asserting that φ is true. By contrast, asserting simple indicative conditionals amounts to making *conditional assertions*, that is to assert the consequent with the proviso that the antecedent is true. The assertion of the consequent is conditional in the sense that it expresses a conditional commitment. In uttering a (simple) conditional sentence, the utterer anticipates in advance that should the antecedent turn out to be false the assertion will be *eo ipso* cancelled. To capture the semantical aspects of conditional assertion, we have to move from the classical logic of sentences to the logic of tri-events. The idea that the proper assertive speech act relative to conditionals is a conditional assertion is not new. It was indeed implicit in Ramsey's and de Finetti's thought in the thirties—Ramsey (1928) 1990: 155 n.; de Finetti (1934) 2006: 103. It was half-heartedly advanced by Quine (1966: 12) and advocated by a few other authors after that—among them von Wright (1957), Belnap (1970), Mackie (1973), Edgington (2007: 176-80), De Rose-Grandy (1999).

The problem arises about the assertion of compounds of conditionals. If the assertion of two simple conditionals consists of two separate conditional assertions, in what sense the conjunction or the disjunction of them is a single conditional assertion? Since several antecedents are involved while conditional assertion involves a single antecedent, it seems that the view according to which the assertion of a tri-event is a conditional assertion seems to be inapplicable. This difficulty is, however, easily solved by theorems 5.4 and 5.14. Since the truth conditions of every tri-event are the same as a simple tri-event (that one may effectively find) and asserting the former amounts to asserting the latter, we may assume that asserting a tri-event is, in a general way, a single conditional assertion.[13] We may remark that in natural language, compounds of conditionals have little use. Admittedly, our theory, like other theories that allow compounds of conditional without restraint, extend the available syntactical combinations of natural language conditionals. This fact is typical of logical theories equipped with recursively defined connectives. Even the language of sentential logic extends the natural use of connectives borrowing from algebra the use of parentheses. Before Leibniz and Boole, logicians did not consider complex sentences built by Boolean connectives, and surely they are foreign to natural language. Nobody would understand the truth conditions of enough complex sentences of classical sentential logic without analysing them by the semantical rules. The situation with conditionals is parallel. If we adopt well-defined semantics, we must resort to the semantical rules to understand the truth conditions of enough complex sentences according to such semantics. Admittedly, this happens (in contrast with classical sentential logic) with much less complex compounds of conditional sentences. In any case, it is a matter of degree, not a fundamental difference.

Since there are several alternative semantics, the problem arises to select, among these alternatives, the one better suited in those few cases in which com-

[13] Here the problem of logical omniscience arises. However, this problem hits the application of every logical theory to knowledge and belief. The present account is no exception.

pound conditionals have a definite meaning in natural language. We will endeavour in this last direction in the next section. In the light of the above considerations, we assume, in the following discussion, that tri-events are simple conditionals with two-valued antecedents and consequents. To simplify further the picture, we also assume that no modal symbol occurs in them.

The idea of a truth-conditional *semantics* based on this view has received little acceptance. Lewis' Triviality Results and McGee's results have played a significant role in this attitude. Independently of these results, many philosophers find it difficult to accept that the pragmatic attitude of asserting a proposition with a caveat has a semantical counterpart in a conditional sentence. We reverse the view. Tri-events come first as logical objects *per se*, whose semantics is well defined in the framework of partial modal logic, and we maintain that the proper way to assert them is by conditional assertions.

However, there is also reluctance in accepting the idea of partial semantics concerning conditionals. Partial semantics, although prominent logicians like Kripke (1975) have used it in other contexts, looks like something "deviant", as it requires a departure from the prevailing view that sentences, including conditionals, are either true or false and *tertium non datur*. Another difficulty is that our approach seems to preclude a unified approach to conditionals, encompassing both indicative and counterfactual conditionals. Indeed, considering conditionals with false antecedents as being neither true nor false, seems *prima facie* to preclude a truth-conditional semantics for counterfactuals.

We insist on claiming that making a conditional assertion of a tri-event φ does not mean asserting that φ is true. One must not confuse the assertion of φ with the assertion of $\uparrow\varphi$, except when φ and $\uparrow\varphi$ are logically equivalent (that is when φ is a two-valued sentence). When one asserts φ is asserting that φ is true under the hypothesis that φ has a truth-value, which depends on the truth-value of a two-valued sentence. If that condition fails, nothing is asserted. This fact, pragmatically, implicates that the author of the assertion does not know the truth-value of the antecedent. In normal circumstances, the utterance of tri-event amounts to the utterance of an epistemic open conditional. The class of open conditionals largely overlaps the class of the so-called indicative conditionals (so that the non-explicit implicature is largely conventional). If, after the utterance of a conditional of this kind, the utterer becomes certain that the antecedent is true, she should be prepared to make a categorical assertion of both the consequent and the antecedent.

We may ask what is the proper speech act if the antecedent of an open conditional turns out to be false. There is no commitment to the truth of the consequent under the condition that the antecedent is false. So, suppose the antecedent turns out to be false. In that case, the *conditional commitment to the truth* of the consequent fails because it fails the condition under which that commitment would be at work according to the conditional assertion. Thus, the corresponding tri-event turns out to have no truth value.

8. Non-simple Conditionals

8.1 Negation

There is only one primitive negation (represented by the connective '$\neg$'). Denying a tri-event is denying its consequent while keeping the antecedent as an explicit supposition. As asserting that φ is a conditional assertion, denying that φ is a conditional denial. The conditional denial of a tri-event φ is just the conditional assertion of its negation. As we must not confuse the (categorical) assertion of $\uparrow\varphi$ with the conditional assertion of φ, we must not confuse the conditional denial of φ — that is the conditional assertion of $\neg\varphi$) — with the (categorical) assertion of $\uparrow\neg\varphi$, that is with the (categorical) assertion that φ is false. The latter logically entails the former without being equivalent to it. Nor we must confuse the act of conditionally denying that φ with categorically denying that φ is true, which is entailed by conditionally denying that φ but does not entail it. Of course, one may categorically assert that $\uparrow \varphi$ (that is, one may assert that φ is true) in response to the conditional assertion of $\neg\varphi$. This assertion entails the conditional denial of the conditional claim. Analogously, if one conditionally asserts that φ, the other party may categorically assert that $\uparrow \neg\varphi$ (that is that φ is false), which entails *a fortiori* conditionally denying that φ, that is conditionally asserting that $\neg\varphi$.

8.2 $\wedge$-Introduction Fails

The simultaneous assertion of two-valued sentences φ and ψ may be represented either by the set $\Gamma = \{\varphi, \psi\}$ or by the conjunction $(\varphi \wedge \psi)$. Indeed, Γ and $(\varphi \wedge \psi)$ share the same set of logical consequences. This result is a consequence of the following two deductive properties of sentential conjunction: $\wedge$-introduction and $\wedge$-elimination. According to $\wedge$-introduction $(\varphi \wedge \psi)$ is a logical consequence of Γ. According to $\wedge$-elimination both φ and ψ are logical consequences of $(\varphi \wedge \psi)$. Now, given two sentences whatsoever of our theory, $\wedge$-elimination holds in a general way. By contrast, if φ and ψ are not both two-valued, then $\wedge$-introduction fails.[14] This fact means that asserting (or believing) two tri-events simultaneously is not the same as asserting (or believing) their conjunction. Consider, for example, two tri-events of the form $(\psi \mid \varphi)$ and $(\psi \mid \neg\varphi)$. In this case the set $\Gamma = \{(\psi \mid \varphi), (\psi \mid \neg\varphi)\}$ may well be consistent or satisfiable, so that one may well assert both tri-events (in the conditional sense explained above). However, from Γ does not follows in general $\chi = (\psi \mid \varphi) \wedge (\psi \mid \neg\varphi)$. Indeed, if φ and ψ are logically independent two-valued sentences, χ is necessarily false.

There is no conjunction for which both the introduction rule and the elimination rule hold (see Adams 1998: 177, Schulz 2009). Intuitively, if φ and ψ are two-valued sentences, conditional asserting both that $(\psi \mid \varphi)$ and that $(\psi \mid \neg\varphi)$ would imply asserting that ψ (asserting that ψ conditionally to both φ and $\neg\varphi$ amounts to unconditionally asserting that ψ). Indeed, it holds that ψ is a logical consequence of $\{(\psi \mid \varphi), (\psi \mid \neg\varphi)\}$. So, in general, a set (rather than conjunction) represents the *deductive* content of two or more tri-events (meant as the set of their

[14] Alternative Sobociński conjunction and disjunction definable in three-valued logic are not suitable in our semantics since they do not satisfy associativity.

logical consequences). In applied logic, we may represent the simultaneous assertion of two or more conditionals by the set of those conditionals. Our semantics perfectly represents the set of the logical consequences of this speech act.

However, if one uses the conjunction instead, the intended meaning of the compound sentence should be determined by the adopted semantics. It is not correct to borrow it mechanically from two-valued sentence logic. If we look at natural language, there is little conjunction out of simultaneous assertion of conditionals. In the light of this consideration, we cannot use an iterative conjunction connective to represent that sentence whose assertion is equivalent to the assertion of both the conjuncts, simply because it does not exist. However, we may well obtain this use of the conjunction resorting to a set of conditional sentences.

On the other hand, the fact that our theory extends the natural semantics of conditionals, adding a "conjunction" of conditionals giving to it a meaning different from simultaneous assertion, is not, in our view, a good reason to reject the present theory as far as compounds of conditionals are concerned (let alone to conclude that compounds of conditionals lack truth conditions). After all, the new conjunction (even if foreign to ordinary language) may turn out to be useful in other contexts (especially in automated reasoning).

8.3 Sets as Disjunctions

In sentential logic, the relationship between disjunction and assertion is not analogous to conjunction and assertion. If φ and ψ express two-valued propositions, then asserting that either φ or ψ obtains does not entail either asserting φ or asserting ψ. One may assert $(\varphi \vee \psi)$ while asserting neither φ nor ψ. The same holds for tri-events. Is there a way to express that we are prepared to assert at least one of two or more tri-events? The answer is yes if the desired "disjunction" is expressed by a set of sentences, being the succedent of a sequent. So, in a sequent $\Gamma \Rightarrow \Delta$, Δ is a set of sentences such that at least one of them must be conditionally asserted to their respective antecedents if one asserts all the elements of Γ. The desired disjunction should be stronger than $\vee$, so that the sequent $(\varphi \vee \psi) \Rightarrow \{\varphi, \psi\}$ should not be valid.

8.4 Logical Consequence for Multiple Conclusions (Structural Sequents)

We have seen that our notion of logical consequence is coextensive with Adams *p*-entailment. Now, Adams developed also a sequent version of his probabilistic logic. Since sequents allow representing conjunction and disjunction in a manner that is different from conjunctions and disjunctions represented by connectives, the problem arises whether it is possible to provide a truth-conditional counterpart of his probabilistic definition coextensive with it. The answer is definite, as the following results will show.

Definition 8.1 (Valid Sequents)

The finite set Δ of sentences is an $\mathcal{M}$-consequence of the finite set of sentences Γ ($\Gamma \underset{\mathcal{M}}{\Rightarrow} \Delta$) iff either

(a) There is a non-empty subset Γ' of Γ such that at every world α in $\mathcal{M}$ there is an element φ of Γ' such that $\overline{\models}^{\mathcal{M}}_{\alpha}\ \varphi$
or

(b) there is a subset Γ' of Γ and a non-empty n-tuple $\Delta' = \langle \varphi_1, \ldots, \varphi_n \rangle$ of elements of Δ such that the following conditions are satisfied:

  (i) for every world α in $\mathcal{M}$ such that no element of Γ' is false at α and at least one element of Γ' is true at α it holds that at least one element of Δ' is true at α;

  (ii) for every world α in $\mathcal{M}$, such that no element of Γ' is false at α and for some i ($1 \leq i \leq n$) either $\models^{\mathcal{M}}_{\alpha}\ \varphi_i$ or $\overline{\models}^{\mathcal{M}}_{\alpha}\ \varphi_i$ while for all $j < i$ (if any) not $\models^{\mathcal{M}}_{\alpha}\ \varphi_j$ and not $\overline{\models}^{\mathcal{M}}_{\alpha}\ \varphi_j$, it holds that $\models^{\mathcal{M}}_{\alpha}\ \varphi_i$;

or

(c) $\Gamma \cup \Delta$ is singular in $\mathcal{M}$.

Definition 8.2 (Sequent *p*-entailment)

The set of sentences Γ *p*-entails the set of sentences Δ in the model $\mathcal{M}$ iff either

(a) for every $\epsilon > 0$ there is a $\delta > 0$ such that every probability function $\mathbf{P}^{\mathcal{M}}$ defined over $\mathcal{M}$ according to which every element $\psi \in \Gamma$ such that $\mathbf{P}^{\mathcal{M}}(\updownarrow \psi) > 0$ has a probability value $\geq \delta$ assigns a probability value $\geq \epsilon$ to at least one element of Δ
or

(b) $\Gamma \cup \Delta$ is singular in $\mathcal{M}$.

Conditions (c) in definition 8.1 and (b) in definition 8.2 were not present in Adams original definition. We have added them because, in our system, singular elements are present but have no probability value. At the same time, in Adams 1986 it is assumed that such conditionals have probability 1 (so that such conditions follow from the other conditions). In Adams 1975 they are not present in the language.

Now we are in the position of proving that our definition of sequent validity is coextensive to Adams' sequent *p*-entailment. To prove this, we resort, like in the case of single conclusion inferences to Adams 1986 and Bamber's work (1994). Then, adapting from Bamber, slightly correcting Adams' original yielding for *p*-entailment for sequents, we define it in the following way:

Definition 8.3 (Sequent Yielding)

Let Γ and Δ be finite sets of sentences of the form $(\psi \mid \chi)$ where χ and ψ are in nqc form and χ is not countervalid. Γ *yields* Δ in $\mathcal{M}$ iff either

(A) Δ is empty and for every element φ of Γ, φ is verified at no world in $\mathcal{M}$ by the truth-assignment $t_\alpha^{\mathcal{M}}$ (as explained above) and is falsified by $t_\alpha^{\mathcal{M}}$ at some world in $\mathcal{M}$
or

(B) Δ is not empty and the following conditions are satisfied:

(i) every truth assignment $t_\alpha^{\mathcal{M}}$ (if any) that does not falsifies any element of Γ and verifies at least one element of Γ verifies at least one element of Δ;

(ii) there is an order $\preccurlyeq$ on the set Δ such that if for every truth-assignment $t_\alpha^{\mathcal{M}}$ there is at least one element of Δ which $t_\alpha^{\mathcal{M}}$ either verifies or falsifies then $t_\alpha^{\mathcal{M}}$ verifies the first of such elements according to the order $\preccurlyeq$ (the preceding ones by $\preccurlyeq$ being neither verified nor falsified by $t_\alpha^{\mathcal{M}}$).

Bamber (1994) proves the following result (which we adapt here to our model-theoretic approach):

Theorem 8.1

Let $\mathcal{M}$ be a model. Γ p-entails Δ in $\mathcal{M}$ iff there is a subset Γ' of Γ and a subset Δ' of Δ such that Γ' yields Δ'.

Theorem 8.1 allows proving in our truth-conditional semantics the coextension of p-entailment and validity for sequents.

Theorem 8.2

Let $\mathcal{M}$ be a model. Let $\Gamma \cup \Delta$ be a finite set of sentences syntactically simple sentences of the form $(\psi \mid \chi)$, where χ and ψ are in nqc form and φ is not countervalid in $\mathcal{M}$. Then Γ p-entails Δ in $\mathcal{M}$ iff it holds that $\Gamma \underset{\mathcal{M}}{\Rightarrow} \Delta$.

Proof

Suppose that Γ p-entails Δ in $\mathcal{M}$. By thorem 8.1 there are two sets $\Gamma' \subseteq \Gamma$ and $\Delta' \subseteq \Delta$ such that Γ' yields Δ'. In the case (a) of definition 8.3, at every world α in $\mathcal{M}$ there is an element φ of Γ' such that $\overline{\models}_\alpha^{\mathcal{M}}$ φ. Now either φ is falsified by the truth-assignment $t_\alpha^{\mathcal{M}}$ or φ is neither falsified nor verified by $t_\alpha^{\mathcal{M}}$ but is falsified at some world β in $\mathcal{M}$. Now, in this case Γ' yields $\emptyset$, so that Γ p-entails Δ in the model $\mathcal{M}$. In the case (b) of definition 8.3, suppose that Γ' and Δ' be n-tuples both satisfying conditions (i)-(iii) of definition 8.3. Suppose that for a certain world α in $\mathcal{M}$ for no element φ of Γ' it holds that $\overline{\models}_\alpha^{\mathcal{M}}$ φ and for at least one element ψ of Γ' it holds that $\models_\alpha^{\mathcal{M}}$ ψ. Every sentence that is not true at α is not verified (and not falsified) by $t_\alpha^{\mathcal{M}}$. Let φ be an element of Γ' such that $\models^{\mathcal{M}}_{\alpha} \varphi$. Since, by hypothesis, for every world γ in $\mathcal{M}$ it holds that not $\overline{\models}^{\mathcal{M}}_{\gamma} \varphi$, φ is a near-tautology. Suppose that φ is not verified by $t^{\mathcal{M}}_{\alpha}$. Then there is a world β at which no element of Γ' is falsified and at which φ is verified. Since φ is true at β, by hypothesis there is a subset Δ'' of Δ' such that at least one element ψ of Δ'' is true at β. Again either ψ is verified at β, which would prove that every truth assignment $t^{\mathcal{M}}_{\alpha}$ that does not falsify any element of Γ' and verifies at least one element of Γ' verifies at least one element of Δ'' or ψ is a near-tautology. But in this last case, there is a world γ in $\mathcal{M}$ at which ψ is verified, and no world in $\mathcal{M}$ at which ψ is falsified. In this case $\{\psi\}$ is yielded by $\emptyset$, which is contained in Δ' so that Γ' yields Δ'. So, if condition (i) is satisfied then either every truth assignment $t^{\mathcal{M}}_{\alpha}$ that does not falsify any element of Γ' and verifies at least one element of Γ' verifies at least one element of Δ' or, in any case, Γ *p*-entails Δ. Assume now that condition (ii) of definition 8.3 is satisfied. Suppose that for every world α in $\mathcal{M}$, no element of Γ' is false at α and for some $i(1 \leq i \leq n)$ either $\models^{\mathcal{M}}_{\alpha} \varphi_i$ or $\overline{\models}^{\mathcal{M}}_{\alpha} \varphi_i$ while for all $j < i$(if any) not $\models^{\mathcal{M}}_{\alpha} \varphi_j$ and not $\overline{\models}^{\mathcal{M}}_{\alpha} \varphi_j$. In the case where for no element φ of Γ', it holds that $\overline{\models}^{\mathcal{M}}_{\alpha} \varphi$, if it holds that not $\models^{\mathcal{M}}_{\alpha} \varphi$ and not $\overline{\models}^{\mathcal{M}}_{\alpha} \varphi$, φ is neither verified nor falsified by $t^{\mathcal{M}}_{\alpha}$. If for some element φ of Γ' it holds that $\models^{\mathcal{M}}_{\alpha} \varphi$, φ is either verified at α or it is a near-tautology in $\mathcal{M}$. in the first case let us consider the first element ψ of Δ' such that it holds either that $\models^{\mathcal{M}}_{\alpha} \psi$ or $\overline{\models}^{\mathcal{M}}_{\alpha} \psi$ while for each preceding sentence χ (if any), according to $\preccurlyeq$ it holds that are neither $\models^{\mathcal{M}}_{\alpha} \chi$ nor $\overline{\models}^{\mathcal{M}}_{\alpha} \chi$. By hypothesis ψ is true at α. Now, in this case ψ is either verified or it is a near-tautology in $\mathcal{M}$. In the first case it holds that there is an order $\preccurlyeq$ on the set $\{\varphi_1, \ldots, \varphi_n\}$ such that if for every truth-assignment $t^{\mathcal{M}}_{\alpha}$ there is at least one element of Δ such that $t^{\mathcal{M}}_{\alpha}$ either verifies or falsifies then $t^{\mathcal{M}}_{\alpha}$ verifies the first of such elements according to the order $\preccurlyeq$ (the preceding ones by $\preccurlyeq$ being neither verified nor falsified by $t^{\mathcal{M}}_{\alpha}$). If ψ is a near-tautology then $\emptyset$ yields $\{\psi\}$, so that Γ *p*-entails Δ. Let Γ'' be the set of the elements of Γ which are not near-tautologies. Every element φ of Γ'' (which may be empty) such that it holds that $\models^{\mathcal{M}}_{\alpha} \varphi$ is verified at α by $t^{\mathcal{M}}_{\alpha}$. But in this case, by what has been said, Γ'' yields Δ, so that Γ *p*-entails Δ. The case (b) in definition 8.3 is impossible by the conditions of the theorem.

q.e.d.

8.5 Switches Paradox

The switches paradox is due to Adams that formulates it in the following way:

> If switches *A* and *B* are thrown the motor will start. Therefore, either if switch *A* is thrown the motor will start or if switch *B* is thrown the motor will start (Adams 1975: 32).

Nobody would consider this argument compelling. However, if the material conditional is used to formalise it, it turns out to be valid. So, material conditional appears to be unable to deal appropriately in this case. The paradox evaporates if modal theories (strict implication theory or Stalnaker-Lewis theories) are used instead of the material conditional. The problem arises if our theory can solve the paradox.

P. Milne (1997: 224) writes that "with the de Finetti-Goodman-Nguyen conjunction and disjunction conditional assertions/events provide no escape from the switches paradox". That result is true for the original de Finetti's theory equipped with the natural consequence relation based on the lattice order. It would be true also for our theory if the conclusion in disjunctive form were expressed using $\vee$ (rather than a set in the succedent of a sequent.) However, with this last move, the switches paradox is easily solved. In fact, the sequent $\{(\chi \mid (\varphi \wedge \psi))\}, \Gamma \underset{\mathcal{M}}{\Rightarrow} \Delta, \{\chi \mid \varphi, \chi \mid \psi\}$ is not valid. Notice that this sequent is also not p-valid so that Adams theory offers a solution to the switches paradox as well. Our solution is not in contrast with Adams' solution since our notion of logical consequence is coextensive with p-entailment. Rather, it provides truth-conditional foundations of Adams logic. Indeed, in our theory, the notion of logical consequence is defined throughout in truth-conditional terms.

8.6 Sequent Antecedents and Succedents concerning Connectives

For representing conjunctions and disjunctions, sets are different from connectives. They cannot be combined freely with the connectives, and recursive formation rules do not govern them. The use of sets makes sense only in an argument where the set of those premises represents a conjunctive premise, and the set of those conclusions represents a disjunctive conclusion. In sharp contrast with standard sequent theory, sequent conjunction and disjunction do not coincide with the respective connectives. For example, concerning $\wedge$ and $\vee$ connectives, the sequent conjunction is weaker than $\wedge$, and the sequent disjunction is stronger than $\vee$. Indeed, the following sequent schemas are valid:

$$\frac{(\phi \wedge \psi) \Rightarrow \chi}{\{\phi, \psi\} \Rightarrow \chi} \qquad\qquad \frac{\chi \Rightarrow \{\phi, \psi\}}{\chi \Rightarrow (\phi \vee \psi)}$$

while the inverse schemas are not

$$\frac{\{\phi, \psi\} \Rightarrow \chi}{(\phi \wedge \psi) \Rightarrow \chi} \qquad\qquad \frac{\chi \Rightarrow (\phi \vee \psi)}{\chi \Rightarrow \{\phi, \psi\}}.$$

8.7 Modus Ponens Fails

Modus ponens schema is not valid in our theory. This invalidity is not a defect of the present theory. There is an important result by McGee (1985) about *modus ponens* which shows that *modus ponens* may not preserve truth concerning iterated conditionals. In light of this result, it is surely not a defect of our theory if *modus ponens* is not valid in it except for simple conditionals. Since we know that every tri-event is logically equivalent to a simple tri-event, translating the sentences in an instance of the *modus ponens*, the schema does not yield necessarily another instance of the *modus ponens* schema. Theorems 5.5 and 5.14 show that the simple counterpart of a tri-event has, in general, a S5 antecedent and a S5 consequent.

However, as McGee's example shows, our intuitions about non-simple conditionals cannot be mechanically borrowed by sentence logic. *Modus ponens* is very important in sentence logic concerning material conditional. However, it would be a fallacy to employ it in inferences involving non-simple conditionals without further constraints that render *modus ponens* a valid inference. It also should be stressed that since valid sentences are essentially two-valued in our theory, one may safely adopt *modus ponens* as a proof-theoretic rule in a possible axiomatic system of tri-events (whose development is part of a future agenda).

8.8 Conditional Excluded Middle

Excluded middle cannot hold in our theory because tri-events are not two-valued sentences, so that $(\varphi \vee \neg\varphi)$ is not a valid schema. From this it follows also that *conditional* excluded middle is not valid. For, if $\varphi = (\chi \mid \psi)$ then $\neg\varphi = (\neg\chi \mid \psi)$, so that also $(\chi \mid \psi) \vee (\neg\chi \mid \psi)$ is not valid. However, in a general way, if φ is not singular (so that it may have a truth-value) $(\varphi \vee \neg\varphi)$ is true at every world. That is, excluded middle (including conditional excluded middle) is a near-tautology. Indeed, every classic sentential tautology is a near-tautology according to our semantics.

When the conclusion of an argument has many conclusions (that is, in applied logic, when the conclusion asserts at least one disjunct in the presence of the premises), Adams proved an interesting result. According to it, sequents, once translated in the language of Lewis' modal theory of conditionals, where a sequent of simple conditionals $\{\varphi_1, \ldots, \varphi_n\} \Rightarrow \{\psi_1, \ldots, \psi_m\}$ is translated as $(\varphi_1 \wedge \cdots \wedge \varphi_n) \rightarrow (\psi_1 \vee \ldots \vee \psi_m)$ are *p*-valid iff their translation is valid in the Lewis' VW system (Adams 1986: 269). Since *p*-entailment is coextensive to our logical consequence or sequent validity, this result also holds in our semantics.

8.9 Import-Export

It is well known that the Stalnaker-Thomason's and Lewis' semantics do not satisfy the import-export law for conditionals. Lycan (2001: 82), commenting on Gibbard 1981 notices that the import-export principle entails the logical equivalence between $(A > (B > A)$ and $((A\&B) > A)$ (where '&' stands for 'and' and '>' stands for the conditional). Moreover, since $((A\&B) > A)$ is considered as valid, $(A > (B > A)$ would be valid as well. However, the following example shows that $(A > (B > A))$ is not always true:

> If Harry runs fifteen miles this afternoon, then if he is killed in a swimming accident this morning, he will run fifteen miles this afternoon.

In this case, A ("Harry runs fifteen miles this afternoon") and B ("he is killed in a swimming accident this morning") are such that their conjunction is impossible, and this logical situation generates the paradox. Now, let us examine this example in the light of our modal theory of tri-events. We discover that the schema $((\psi \mid \varphi) \mid \psi)$, where φ and ψ are two-valued sentences, is *not* always valid, but that the only exception is just when $(\varphi \wedge \psi)$ is impossible. The following schema, if φ and ψ are ordinary sentences, is valid:

$\Diamond(\varphi \wedge \psi) \rightarrow ((\psi \mid \varphi) \mid \psi)$

More generally, the following theorem holds:

Theorem 8.3

Let $\hat{\varphi}$, $\hat{\psi}$, and $\hat{\chi}$ ordinary sentences. Then the following schema is O-valid:
(RLIE) $\Diamond(\hat{\varphi} \wedge \hat{\psi}) \rightarrow (((\hat{\chi} \mid \hat{\psi}) \mid \hat{\varphi}) \leftrightarrow (\hat{\chi} \mid (\hat{\varphi} \wedge \hat{\psi})))$

Proof

Suppose that $\models \Diamond(\hat{\varphi} \wedge \hat{\psi})$. Let $\mathcal{M}$ be any model. Since $\hat{\varphi}$ and $\hat{\psi}$ are, by theorem 4.5 essentially two-valued sentences, this means that there is world α in $\mathcal{M}$ it holds that $\models^{\mathcal{M}}_{\alpha} (\hat{\varphi} \wedge \hat{\psi})$, and therefore that $\models^{\mathcal{M}}_{\alpha} \hat{\varphi}$ and $\models^{\mathcal{M}}_{\alpha} \hat{\psi}$. Under this assumption and under theorem 4.3, we have to prove:

(a) if it holds that $\models^{\mathcal{M}}_{\alpha} ((\hat{\chi} \mid \hat{\psi}) \mid \hat{\varphi})$ then $\models^{\mathcal{M}}_{\alpha} (\hat{\chi} \mid (\hat{\varphi} \wedge \hat{\psi}))$;

(b) If it holds that $\models^{\mathcal{M}}_{\alpha} (\hat{\chi} \mid (\hat{\varphi} \wedge \hat{\psi}))$ then $\models^{\mathcal{M}}_{\alpha} ((\hat{\chi} \mid \hat{\psi}) \mid \hat{\varphi})$;

(c) If it holds that $\overline{\models}^{\mathcal{M}}_{\alpha} ((\hat{\chi} \mid \hat{\psi}) \mid \hat{\varphi})$ then $\overline{\models}^{\mathcal{M}}_{\alpha} (\hat{\chi} \mid (\hat{\varphi} \wedge \hat{\psi}))$;

(d) If it holds that $\overline{\models}^{\mathcal{M}}_{\alpha} (\hat{\chi} \mid (\hat{\varphi} \wedge \hat{\psi}))$ then $\overline{\models}^{\mathcal{M}}_{\alpha} ((\hat{\chi} \mid \hat{\psi}) \mid \hat{\varphi})$;

(a) Let $\mathcal{M}$ be any model and let α any world in $\mathcal{M}$. Suppose that $\models^{\mathcal{M}}_{\alpha} ((\hat{\chi} \mid \hat{\psi}) \mid \hat{\varphi})$. By definition 2.4, either (j) $\models^{\mathcal{M}}_{\alpha} (\hat{\chi} \mid \hat{\psi})$ and $\models^{\mathcal{M}}_{\alpha} \hat{\varphi}$ or (jj) there is a world β in $\mathcal{M}$ at which it holds that $\models^{\mathcal{M}}_{\beta} (\hat{\chi} \mid \hat{\psi})$ and $\models^{\mathcal{M}}_{\beta} \hat{\varphi}$ and at no world γ in $\mathcal{M}$ it holds that $\models^{\mathcal{M}}_{\gamma} \hat{\varphi}$ and $\overline{\models}^{\mathcal{M}}_{\gamma} (\hat{\chi} \mid \hat{\psi})$. In the case (j), by definition 2.4, there are two sub-cases: (j1) $\models^{\mathcal{M}}_{\alpha} \hat{\chi}$ and $\models^{\mathcal{M}}_{\alpha} \hat{\psi}$, and (j2) there is a world β in $\mathcal{M}$ at which it holds that $\models^{\mathcal{M}}_{\beta} \hat{\chi}$ and $\models^{\mathcal{M}}_{\beta} \hat{\psi}$ and at no world γ in $\mathcal{M}$ it holds that $\models^{\mathcal{M}}_{\gamma} \hat{\psi}$ and $\overline{\models}^{\mathcal{M}}_{\gamma} \hat{\chi}$. In the case (j1), by definition 2.4 it holds that $\models^{\mathcal{M}}_{\alpha} (\hat{\chi} \mid (\hat{\varphi} \wedge \hat{\psi}))$, since it holds that $\models^{\mathcal{M}}_{\alpha} \hat{\varphi}$, $\models^{\mathcal{M}}_{\alpha} \hat{\psi}$, and $\models^{\mathcal{M}}_{\alpha} \hat{\chi}$. In the case (j2) since, by hypothesis, there is a world β in $\mathcal{M}$ at which $\models^{\mathcal{M}}_{\beta} (\hat{\varphi} \wedge \hat{\psi})$ and at no world γ in $\mathcal{M}$ it holds that $\models^{\mathcal{M}}_{\gamma} \hat{\psi}$ and $\overline{\models}^{\mathcal{M}}_{\gamma} \hat{\chi}$. *A fortiori* at no world γ in $\mathcal{M}$ it holds that $\models^{\mathcal{M}}_{\gamma} (\hat{\varphi} \wedge \hat{\psi})$ and $\overline{\models}^{\mathcal{M}}_{\gamma} \hat{\chi}$. It follows that $\models^{\mathcal{M}}_{\alpha} (\hat{\chi} \mid (\hat{\varphi} \wedge \hat{\psi}))$. In the case (jj), there is no world γ in $\mathcal{M}$ at which $\models^{\mathcal{M}}_{\gamma} \hat{\varphi}$ and $\overline{\models}^{\mathcal{M}}_{\gamma} (\hat{\chi} \mid \hat{\psi})$. In this case, there is no world γ at which $\models^{\mathcal{M}}_{\gamma} \hat{\varphi} \wedge \hat{\psi}$ and $\overline{\models}^{\mathcal{M}}_{\gamma} \hat{\chi}$, while there is a world β in $\mathcal{M}$ at which $\models^{\mathcal{M}}_{\beta} \hat{\varphi} \wedge \hat{\psi}$ and $\models^{\mathcal{M}}_{\beta} \hat{\chi}$. By definition 2.4 it holds that $\models^{\mathcal{M}}_{\alpha} (\hat{\chi} \mid (\hat{\varphi} \wedge \hat{\psi}))$.

(b) Let $\mathcal{M}$ be any model and let α any world in $\mathcal{M}$. Suppose that $\models^{\mathcal{M}}_{\alpha} (\hat{\chi} \mid (\hat{\varphi} \wedge \hat{\psi}))$. By definition 2.4, either (j) $\models^{\mathcal{M}}_{\alpha} \hat{\chi}$ and $\models^{\mathcal{M}}_{\alpha} (\hat{\varphi} \wedge \hat{\psi})$ or (jj) there is a world β in $\mathcal{M}$ at which it holds that $\models^{\mathcal{M}}_{\beta} \hat{\chi}$ and $\models^{\mathcal{M}}_{\beta} (\hat{\varphi} \wedge \hat{\psi})$ and at no world γ it holds that $\models^{\mathcal{M}}_{\gamma} (\hat{\varphi} \wedge \hat{\psi})$ and $\overline{\models}^{\mathcal{M}}_{\gamma} \hat{\chi}$. In the case (j), it holds that $\models^{\mathcal{M}}_{\alpha} \hat{\chi}$, $\models^{\mathcal{M}}_{\alpha} \hat{\varphi}$, and $\models^{\mathcal{M}}_{\alpha} \hat{\psi}$, so that, by definition 2.4 it holds that $\models^{\mathcal{M}}_{\alpha} ((\hat{\chi} \mid \hat{\psi}) \mid \hat{\varphi})$. In the case (jj), there is a world β in $\mathcal{M}$ at which $\models^{\mathcal{M}}_{\beta} \hat{\chi}$,

$\models^{\mathcal{M}}_{\beta} \hat{\psi}$, and $\models^{\mathcal{M}}_{\beta} \hat{\varphi}$ and at no world γ in $\mathcal{M}$ it holds that $\models^{\mathcal{M}}_{\gamma} \hat{\psi}$, $\models^{\mathcal{M}}_{\gamma} \hat{\varphi}$, and $\overline{\models}^{\mathcal{M}}_{\gamma} \hat{\chi}$. By definition 2.4 it holds that $\models^{\mathcal{M}}_{\alpha} ((\hat{\chi} \mid \hat{\psi}) \mid \hat{\varphi})$.

(c) Let $\mathcal{M}$ be any model and let α any world in $\mathcal{M}$. Suppose that $\overline{\models}^{\mathcal{M}}_{\alpha} ((\hat{\chi} \mid \hat{\psi}) \mid \hat{\varphi})$. By definition 2.4, either (j) $\overline{\models}^{\mathcal{M}}_{\alpha} (\hat{\chi} \mid \hat{\psi})$ and $\models^{\mathcal{M}}_{\alpha} \hat{\varphi}$ or (jj) there is a world β in $\mathcal{M}$ at which it holds that $\overline{\models}^{\mathcal{M}}_{\beta} (\hat{\chi} \mid \hat{\psi})$ and $\models^{\mathcal{M}}_{\beta} \hat{\varphi}$ and at no world γ in $\mathcal{M}$ it holds that $\models^{\mathcal{M}}_{\gamma} \hat{\varphi}$ and $\models^{\mathcal{M}}_{\gamma} (\hat{\chi} \mid \hat{\psi})$. In the case (j), by definition 2.4, there are two sub-cases: (j1) $\overline{\models}^{\mathcal{M}}_{\alpha} \hat{\chi}$ and $\models^{\mathcal{M}}_{\alpha} \hat{\psi}$, and (j2) there is a world β in $\mathcal{M}$ at which it holds that $\overline{\models}^{\mathcal{M}}_{\beta} \hat{\chi}$ and $\models^{\mathcal{M}}_{\beta} \hat{\psi}$ and at no world γ in $\mathcal{M}$ it holds that $\models^{\mathcal{M}}_{\gamma} \hat{\psi}$ and $\models^{\mathcal{M}}_{\gamma} \hat{\chi}$. In the case (j1), by definition 2.4 it holds that $\overline{\models}^{\mathcal{M}}_{\alpha} (\hat{\chi} \mid (\hat{\varphi} \wedge \hat{\psi}))$, since it holds that $\models^{\mathcal{M}}_{\alpha} \hat{\varphi}$, $\models^{\mathcal{M}}_{\alpha} \hat{\psi}$, and $\overline{\models}^{\mathcal{M}}_{\alpha} \hat{\chi}$. In the case (j2) since, by hypothesis, there is a world β in $\mathcal{M}$ at which $\models^{\mathcal{M}}_{\beta} (\hat{\varphi} \wedge \hat{\psi})$ and at no world γ in $\mathcal{M}$ it holds that $\models^{\mathcal{M}}_{\gamma} \hat{\psi}$ and $\models^{\mathcal{M}}_{\gamma} \hat{\chi}$. *A fortiori* at no world γ in $\mathcal{M}$ it holds that $\models^{\mathcal{M}}_{\gamma} (\hat{\varphi} \wedge \hat{\psi})$ and $\models^{\mathcal{M}}_{\gamma} \hat{\chi}$. It follows that $\models^{\mathcal{M}}_{\alpha} (\hat{\chi} \mid (\hat{\varphi} \wedge \hat{\psi}))$. In the case (jj) there is no world γ in $\mathcal{M}$ at which $\models^{\mathcal{M}}_{\gamma} \hat{\varphi}$ and $\models^{\mathcal{M}}_{\gamma} (\hat{\chi} \mid \hat{\psi})$. In this case, there is no world γ at which $\models^{\mathcal{M}}_{\gamma} \hat{\varphi}$, $\models^{\mathcal{M}}_{\gamma} \hat{\psi}$ and $\models^{\mathcal{M}}_{\gamma} \hat{\chi}$, while there is a world β in $\mathcal{M}$ at which $\models^{\mathcal{M}}_{\beta} \hat{\varphi}$, $\models^{\mathcal{M}}_{\beta} \hat{\psi}$ and $\overline{\models}^{\mathcal{M}}_{\beta} \hat{\chi}$. By definition 2.4 it holds that $\overline{\models}^{\mathcal{M}}_{\alpha} (\hat{\chi} \mid (\hat{\varphi} \wedge \hat{\psi}))$.

(d) Let $\mathcal{M}$ be any model and let α any world in $\mathcal{M}$. Suppose that $\overline{\models}^{\mathcal{M}}_{\alpha} (\hat{\chi} \mid (\hat{\varphi} \wedge \hat{\psi}))$. By definition 2.4, either (j) $\overline{\models}^{\mathcal{M}}_{\alpha} \hat{\chi}$ and $\models^{\mathcal{M}}_{\alpha} (\hat{\varphi} \wedge \hat{\psi})$ or (jj) there is a world β in $\mathcal{M}$ at which it holds that $\overline{\models}^{\mathcal{M}}_{\beta} \hat{\chi}$ and $\models^{\mathcal{M}}_{\beta} (\hat{\varphi} \wedge \hat{\psi})$ and at no world γ it holds that $\models^{\mathcal{M}}_{\gamma} (\hat{\varphi} \wedge \hat{\psi})$ and $\models^{\mathcal{M}}_{\gamma} \hat{\chi}$. In the case (j), it holds that $\overline{\models}^{\mathcal{M}}_{\alpha} \hat{\chi}$, $\models^{\mathcal{M}}_{\alpha} \hat{\varphi}$, and $\models^{\mathcal{M}}_{\alpha} \hat{\psi}$, so that, by definition 2.4 it holds that $\overline{\models}^{\mathcal{M}}_{\alpha} ((\hat{\chi} \mid \hat{\psi}) \mid \hat{\varphi})$. In the case (jj), there is a world β in $\mathcal{M}$ at which $\overline{\models}^{\mathcal{M}}_{\beta} \hat{\chi}$, $\models^{\mathcal{M}}_{\beta} \hat{\psi}$, and $\models^{\mathcal{M}}_{\beta} \hat{\varphi}$ and at no world γ in $\mathcal{M}$ it holds that $\models^{\mathcal{M}}_{\gamma} \hat{\psi}$, $\models^{\mathcal{M}}_{\gamma} \hat{\varphi}$, and $\models^{\mathcal{M}}_{\gamma} \hat{\chi}$. By definition 2.4 it holds that $\overline{\models}^{\mathcal{M}}_{\alpha} ((\hat{\chi} \mid \hat{\psi}) \mid \hat{\varphi})$.

q.e.d.

So, as far as Import-Export Law is concerned, our theory appears to be better suited than Stalnaker-Thomason's and Lewis' semantics. It avoids the difficulty raised by Lycan while keeping the law, albeit in a properly restricted form.

9. Conclusion

In this paper, I have presented, moving from de Finetti's idea about the so-called tri-events, a new modal semantics of conditionals. This semantics encompasses

Adams' probabilistic semantics and is coextensive to it as simple conditionals are concerned. Compounds of conditionals are logically equivalent to simple conditionals. We provide generalised probability axioms, which come down to the usual axioms for finite probability when only ordinary sentences are involved. The new theory bypasses Lewis' Triviality Results since the probability of a conditional is always the conditional probability of the consequent, given the antecedent. This result is possible because (a) non-simple conditionals obey the generalised axioms of probability, not to the standard axioms (the underlying algebraic structure being a lattice but not a Boolean algebra), and (b) we drop the central premise underlying the triviality results, according to which conditionals are two-valued sentences.

The Kripke-style modal semantics presented in this paper is a *partial semantics* so that there are distinct conditions for truth and falsehood. Moreover, in this theory, binary connectives, except material implication, which returns a two-valued sentence, have a *modal import*: their semantics depends on the totality of worlds. However, no modal import is present when mutually independent sentences are involved.

Adams' *p*-entailment may be applied unmodified to probability functions defined over the lattice of tri-events. Every sentence is logically equivalent to a simple conditional, where both antecedent and consequent are S5 sentences. We defined probability concerning a model $\mathcal{M}$. The probability of a tri-event is always equal to the ratio between the probability of two S5 sentences, provided the latter probability is greater than 0. There is no difficulty in dealing with the probability of S5 sentences since modal sentences and their negations (i.e. sentences of the form $\Diamond\varphi$, $\Box\varphi$, $\neg\Diamond\varphi$, and $\neg\Box\varphi$) in a single model behave like tautologies if true and like contradictions if false so that they have probability 1 or 0 at every world in a given model. Now, we have defined a truth-conditional consequence relation that is coextensive to *p*-entailment. This result challenges Adams tenet that conditionals have no truth conditions. Moreover, the probability of a tri-event is always the *conditional expectation* of the consequent given the antecedent. We also challenge Adams-like views that understand the probability of conditionals as the degree of assertability or acceptability without any connection with truth-values.

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