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Argumenta is the official journal of the Italian Society for Analytic Philosophy (SIFA). It was founded in 2014 in response to a common demand for the creation of an Italian journal explicitly devoted to the publication of high quality research in analytic philosophy. From the beginning *Argumenta* was conceived as an international journal, and has benefitted from the cooperation of some of the most distinguished Italian and non-Italian scholars in all areas of analytic philosophy.

Contents

Editorial

The present issue of *Argumenta* opens with a Special Issue entitled *Conditionals and Probability*, edited by Alberto Mura. I do not think it is an exaggeration to say that—without pretending to be totally exhaustive—it is a remarkable advance in a field relevant to the philosophy of language, the philosophy of science, philosophical logic, psychology, logic, linguistics, and AI.

For one thing, some of the articles represent a step towards a unified theory for conditional sentences, in that they try to encompass various kinds of conditionals in a single framework. And, for another, some articles stress the importance of the contribution Bruno de Finetti made to the topic towards the end of the 1920s, especially in the light of the most recent experimental research which has shown the connection between de Finetti's notion of tri-events and everyday reasoning involving conditionals.

After the Special Issue, the section of Book Reviews rounds off the number. In this section, readers will find a careful assessment of three very interesting recent books—*Handbook of Legal Reasoning and Argumentation* edited by G. Bongiovanni, G. Postema, A. Rotolo, G. Sartor, C. Valentini, and D. Walton, *Logic from Kant to Russell: Laying the Foundation for Analytic Philosophy* edited by S. Lapointe, and *Kant's Critical Epistemology*: *Why Epistemology Must Consider Judgment First* by K. Westphal.

Finally, I would like to thank all the colleagues who have acted as external referees, the Assistant Editors, the Editor of the Book Reviews, and the mem-

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bers of the Editorial Board. All of them have been very generous with their advice and suggestions.

All the articles appearing in *Argumenta* are freely accessible and freely downloadable; therefore it only remains to wish you:

Buona lettura!

Massimo Dell'Utri
Editor-in-Chief

Argumenta 6, 2 (2021)
Special Issue

Conditionals and Probability

Edited by

Alberto Mura

The Journal of the Italian Society for Analytic Philosophy

Contents

Conditionals and Probability: Introduction

Alberto Mura

University of Sassari

This Special Issue aims to take stock of the philosophical debate about the relationship between conditionals and probabilities.

Three of the contributions (Nulvesu, Baratgin, Mura) revolve around de Finetti's truth-conditional theory of tri-events. The paper by Gilio (who was a student of his) and Sanfilippo concentrates on de Finetti's view of probabilistic semantics as grounded on the notion of coherence. Two papers (Bradley and Schulz) focus on the relationship between probability and counterfactual conditionals. Finally, the paper by Crupi and Iacona uses the notion of probabilistic relevance to capture the relationship between antecedent and consequent of a conditional.

The issue opens with Bradley's paper, which presents a range of suggestions within the framework of the suppositional conception of conditionals. The most intriguing, in my view, is the distinction between *evidential* suppositions and *interventional* suppositions. Evidential suppositions are hypotheses that, if true, should be added to the totality of the accepted information for determining the degrees of belief or epistemic probability. Interventional suppositions are assumed to be true as if they were due to an intervention external to the set of causal relationships that naturally hold among the events under consideration. Thanks to this distinction, Bradley can give a new characterization of the distinction between indicative and subjunctive (also called *counterfactual*) conditionals. Roughly speaking, the antecedent is an evidentiary supposition in indicative conditionals, while the antecedent is an interventional supposition in subjunctive conditionals.This approach allows Bradley to maintain that Adams' Thesis (that the probability of a conditional is the conditional probability of the consequent given the antecedent) applies to both indicative and subjunctive conditionals. This tenet suggests a unified suppositional view of indicative and subjunctive conditionals, which Adams rejected.

Schulz's paper shows that fresh problems arise about conditionals and probability if we also consider the notion of knowledge. He discusses a puzzle originally introduced by Rothschild and Spectre, and shows that it may be split into two parts, each of which can be resolved separately by the application of a single solution, thus proving the intimate connection among conditionals, probability, and knowledge.

Crupi and Iacona start from the idea (recently proposed by Bouven) that a conditional statement *C if A* is the more assertable, the more its antecedent *A*

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provides evidential support to the consequent *C* so that it *increases* its probability. This view is at odds with Adams' Thesis, according to which the degree of assertability of a *C if A* is just the conditional probability $Pr(C \mid A)$ of the consequent *C* given the antecedent *A*,irrespective of whether *A* supports *C* or not. Crupi and Iacona define a new probabilistic logic with its probabilistic syntax and semantics. Their semantics defines a validity notion that shares some properties with Adams' p-entailment but diverges from it in other respects. For example, like Adams' language, their language does not contain compounds of conditionals, but unlike Adams', it has a recursive formation rule for the negation connective. The authors compare their logic with Adams' logic in detail, arguing that theirs is better suited in modelling a logic for indicative conditionals than is his (or, indeed, Douven's). The reader will assess this theory for herself, but, in any case, it deserves very serious consideration.

Nulvesu's paper represents a transition between the foregoing papers and those that follow and that are focused on the logic of de Finetti's tri-events. The paper aims to put de Finetti's theory of tri-events (dating from the first decades of the 20th century) in the context of the contemporary debates about conditionals. Nulvesus' paper is also useful to readers looking for an overview of the contemporary debates about conditionals, including Stalnaker-Lewis truth-conditionals views and Adams' views.

De Finetti's ideas permeate the paper by Gilio and Sanfilippo. However, they depart from de Finetti's truth-conditional tri-events and take a different direction, based on the de Finettian notion of coherence, namely, the idea that propositions or events are *random quantities* and conditional expectations. On this picture, it is not clear what the values 1 and 0 represent. They may be understood as genuine truth-values (albeit in non-realistic terms), as in the theory of tri-events, or as simply "values" or, to use Adams' phrase, as "ersatz truth-values". Gilio advanced the core ideas on which the present contribution is based in joint authorship with Romano Scozzafava.[1] The fundamental notion is that of a bet. However, de Finetti himself defined coherence in that context and extended it to random variables in general. De Finetti also introduced the idea of a conditional bet on an event C given an event A, characterized as a bet such that the bettor wins if both A and C occur, loses if A occurs, but C does not, and is void (or null) if A fails to occur. If the fair betting quotient of this bet is *p*, so that the fair price of the offer "1 if A and C, 0 if A and not C, and *p* if not A" has expected value *p*, Gilio and Scozzafava took the real number *p* as the (quasi-semantical—the qualification is mine) value of the conditional event C | A.

Gilio and Sanfilippo have developed this idea much more fully, but unfortunately, philosophers have little knowledge of their valuable work, despite the great interest that it presents from the philosophical point of view. Therefore, the publication of the essay in this issue aims also to disseminate their work among philosophers given its focus on the aspects that are most relevant to the debate on conditionals: compounds of conditionals and iterated conditionals. Their theory satisfies in a general way the Adams Thesis. So it escapes *Lewis' Triviality Results*. The theory of Gilio and Sanfilippo attributes a finite number of values to

[1] In the same year (1994), independently, Robert Stalnaker and Richard Jeffrey proposed a similar approach, although within a Kripke-style model semantics (cf. Stalnaker and Jeffrey 1994).

a conditional. In principle, this theory deals with conditionals of any complexity, even if the number of quasi-semantic values of a complex sentence grows with the number of distinct atomic sentences it contains. The probability of a complex event remains the expected value of its quasi-semantical values. A very remarkable feature of this theory is that its conjunction connective for simple conditional events satisfies, for *p*-entailment, both the introduction and elimination rules, which is a property that none of the truth-conditional theories satisfying the Adams' Thesis can satisfy.

I have qualified the values of Gilio-Sanfilippo conditional events as random quantities as 'quasi-semantic' so that the values 1 and 0 turn out to be ersatz truth-values rather than genuine truth values. Thus, there is no truth-conditional theory that supports the truth-conditional interpretation. From a philosophical point of view, one must consider the Gilio-Sanfilippo theory as a generalisation of Adams's theory, which postulates that conditional events (except the degenerate case in which they come down to ordinary events) are devoid of truth-value but have a degree of assertability, expressed by their probability. Of course, when only simple conditional events are involved, the Gilio-Sanfilippo theory coincides with Adams's theory.

Baratgin's paper is fundamentally a very important historical study of the origin of de Finetti's theory of tri-events. Baratgin's research antedates de Finetti's introduction of tri-events to 1928, while it was generally supposed that de Finetti invented them in the 1930s. *Prima facie*, this seems to be just a historical detail. It is not. To understand why one has to recall that the late Richard Jeffrey, after reading de Finetti's *Probabilismo* (1931) maintained that, *at that time*, de Finetti was, like himself, a *radical probabilist*. Radical probabilism is the doctrine according to which probability is a *primitive notion* and that one may characterise it without appealing to events that are considered certain. So, according to this doctrine, probability comes before truth. Was de Finetti a radical probabilist? In *De Finetti's Radical Probabilism*, Jeffrey writes:

> De Finetti's probabilism is "radical" in the sense of going all the way down to the roots: he sees probabilities as ultimate forms of judgment which need not be based on deeper all-or-none knowledge (Jeffrey 1993: 264).

I discussed the matter with the late Horacio Árlo Costa via email and face-to-face in 2011 in Konstanz. According to Árlo Costa "this view is mistaken. There is plenty of evidence in de Finetti's writings that de Finetti did appeal to background certainties, which he used to define the notion of possibility. Finally, probability is distributed over this `field of possibility'". Since both Jeffrey's and Árlo Costa's views are well supported by textual evidence, in continuing our discussion, Árlo Costa and I agreed on a reading of de Finetti as a radical probabilist when he wrote *Probabilismo*. However, we concluded that he changed his mind later, thus following Jeffrey's reverse intellectual pathway (who had abandoned the positions of Carnap to reach his radical probabilism). Now, de Finetti composed *Probabilismo* in 1928. Baratgin's paper shows that de Finetti elaborated his theory of tri-events in that year. Since this theory is based on truth-values and inspired by the betting situation, where the outcome is the paradigm of a certain event that decides whether the bet is won or lost, rethinking this matter, I concluded, in the light of Baratgin's paper, that this puzzle might be solved in the face of all the aspects of the complex de Finetti's epistemology (albeit not explicitly elaborated). His sophisticated philosophy was influenced by Hume, Italian pragmatism, instrumentalism, operationalism, Machian philosophy, Vaihinger's "*als ob*" philosophy, and logical positivism. However, one can identify his epistemology with none of these views. Indeed, reading *L'invenzione della verità* (a philosophical essay that de Finetti wrote as a reaction to Carnap's *Der logische Aufbau der Welt* [1928]), it appears clear that one may combine the two conflicting views in the light of these considerations. De Finetti was a radical probabilist for all his scientific life. However, his radicalism is compatible with a constructive attitude to building a fairly idealised theory that gives rise to abstract notions, including truth and probability and the notions of event and conditional event or tri-event. This view is strongly influenced by Carnap's *Aufbau*, although his conclusions are completely at odds with Carnap's view, according to which the building of the "world" and knowledge rests on a solid foundation:

> We see that everything is built on quicksand, although obviously, one tries to place the pillars on the relatively less dangerous points (de Finetti [1933] 2006: 145, English translation is mine).[2]

I hope to explain in more detail my reconstruction of de Finetti's complex epistemological views elsewhere. I want to emphasise here that Baratgin's paper sheds much light on this topic, suggesting that de Finetti never changed his views throughout his scientific life, especially on probability, events, and tri-events. Moreover, Baratgin's paper usefully compares de Finetti's theory with some more recent theories and developments.

My own paper presents a modified version of the truth-conditional theory of tri-events. I begin with the consideration that one cannot equip de Finetti's original theory with a notion of logical consequence in agreement with Adams' logic. Earlier attempts to reconcile tri-events and Adams' logic, proposed by myself or others, all suffer from the defect of having recourse to modal conditions in the definition of logical consequence. As a result, valid formulas cannot be instances of general schemas formulated by metalinguistic variables. Furthermore, those attempts inevitably attribute a special character[3] to basic or atomic statements, a remnant of Wittgenstein's *Tractatus* that de Finetti resolutely rejected (and that I, too, reject).

The semantics presented here is free from these difficulties. However, throughout a modal theory (formulated in Kripke's style), it is more exactly a generalization of a partial logic version of the S5 system. It extends Adams' theory to iterated and compound conditionals of any complexity so that logical consequence and Adams' *p*-entailment always coincide. The main philosophical aim of my contribution is to refute Adams' view that indicative conditionals al-

[2] De Finetti's views seem more like Popper's view as expressed in *The Logic of Scientific Discovery*: "Science does not rest upon solid bedrock. The bold structure of its theories rises, as it were, above a swamp. It is like a building erected on piles" (Popper [1935] 1992: 94).

[3] Cf.: "If we do not choose to ignore the way in which S has been derived from the basis B , the possibility arises that we could single out certain events as being somewhat *special*: for example belonging to the basis, or logically expressible in terms of a finite or countable number of basic elements" (de Finetti [1970] 1975: 271).

ways lack truth-values. My theory also satisfies Adams' equation so that it bypasses *Lewis' Triviality Results*.

I wish to thank Massimo Dell'Utri, editor-in-chief of *Argumenta*, for inviting me to edit this special issue. I also thank Richard Davies for his stylistic suggestions regarding this Introduction. My gratitude especially goes to the authors who contributed to this issue. I also thank the referees who have lent their competent contributions to guaranteeing the high quality of all the papers. Finally, I would like to thank the *Argumenta* staff, who took on the task of designing this issue according to the editorial criteria of the journal. I trust that those interested in the relationship between conditionals and probabilities will appreciate all the effort to deliver this special issue.

References

Carnap, R. 1928, *Der logische Aufbau der Welt*, Berlin-Schlachtensee: Weltkreis.

de Finetti, B. 1931, *Probabilismo: Saggio critico sulla teoria delle probabilità e il valore della scienza*, Napoli: Perrella.

de Finetti, B. (1933) 2006, *L'invenzione della verità*, Milano: Cortina.

de Finetti, B. (1970) 1975, *Theory of Probability*, Vol. 2, Chichester: Wiley.

Jeffrey, R. 1993, "De Finetti's Radical Probabilism", in Monari, P. and Cocchi, D. (eds.), *Probabilità e induzione. Induction and Probability*, Bologna: CLUEB, 263-75.

Popper, K.R. (1935) 1992, *The Logic of Scientific Discovery*, London: Routledge.

Stalnaker, R. and Jeffrey, R. 1994, "Conditionals as Random Variables", in Eells, E. and Skyrms, B. (eds.), *Probability and Conditionals: Belief Revision and Rational Decision*, Cambridge:Cambridge University Press, 1994, 31-46.

Probabilities of Counterfactuals

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Abstract

The subjective probability of a subjunctive conditional is argued to be equal to the expected conditional credence in its consequent, given the truth of its antecedent, of an 'expert': someone who reasons faultlessly and who, at each point in time, is as fully informed about the state of the world as it is possible to be at that time.

Keywords: Conditionals, Suppositions, Probability, Subjunctives, Counterfactuals, Chances.

1. The Suppositional Theory of Conditionals

The Suppositional Theory says that one should believe a conditional to the degree that one believes its consequent to be true on the supposition that its antecedent is or were true, a claim known as the Ramsey Test hypothesis.[1] The theory is widely reckoned to do a good job in explaining the patterns in our attitudes to conditionals and the linguistic behaviour that manifest them: our assertions and denials of different conditionals, the inferences we make with them, and so on. There has been much controversy around the consistency of the Suppositional theory with truth conditional semantics, but I shall set this issue aside.[2] My concern will instead be to give the theory enough additional content to deal with the challenge of explaining how our attitudes to conditionals vary with their morphology; in particular, with mood and tense.

To get us started, let's look at a set of conditionals with common component sentences, but varying morphology, concerning Jim, a canny investor who very rarely loses money, and a potential investment in ostrich farming futures, a fashionable financial instrument, the market for which many consider to be a bubble. Suppose that at time t_0 Jim must decide whether or not to invest in ostrich futures. Jim almost always makes money from his investments so I am inclined to believe that:

[1] Versions of the Suppositional theory have been proposed by, amongst other, Adams (1965, 1975), Bradley (2017), Edgington (1995, 2004, 2008), Skyrms (1981), Stalnaker (1984) and McGee (1987). The origins of the Ramsey Test lie in Ramsey 1929.
[2] For the record I believe this controversy to be resolved: there is no inconsistency.

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(1) If Jim does invest in ostrich futures, then he will make a packet.

On the other hand, from my less well-informed perspective, speculation on ostrich futures looks like a good way to lose a lot of money. So, I am inclined to believe that he won't make the investment but that:

(2) If Jim were to invest in ostrich futures, he would make a loss.

Suppose that by time t_1 Jim has made his decision but I have not learnt what it is or how ostrich futures have performed. Given this, I will continue to regard it as probable that he didn't make the investment but that:

(3) If Jim did invest in ostrich futures, he made a packet.

(4) If Jim had invested in ostrich futures, he would have lost a packet.

I subsequently learn (at time t_2) how ostrich futures have performed. Suppose that they have performed badly. Then I would continue to believe (4) and that Jim didn't make the investment, but that, contrary to (3), he lost a packet if he did. Suppose instead that by t_2 I know that ostrich futures have performed well. Then I would continue to believe (3) and that, contrary to (4), he would have made a packet if he had invested in ostrich futures.

The challenge is to explain this pattern of beliefs with the Suppositional theory: in particular why our attitudes to the indicative conditional and the corresponding subjunctive differ at t_0 and t_1 but not at t_2. These sentences by no means exhibit all morphological variation in conditional sentences of potential interest, but they serve to illustrate four important classes: the forward-looking or future-oriented indicatives and subjunctives ((1) and (2) respectively) and the backward-looking or past-oriented indicatives and subjunctives ((3) and (4) respectively). So getting them right is an important step in filling out the theory.

A terminological note. Subjunctive conditionals are often termed 'counterfactuals' in the philosophical literature, on the grounds that the subjunctive mood is typically used to convey the belief that the antecedent is false. But, as many authors have observed, this implication holds only for the backward-looking subjunctives. Moreover, believing the antecedent of an indicative false doesn't preclude asserting it, let along rendering it senseless (as in 'If he is in the room, then I must be blind'). On the other hand, the indicative-subjunctive distinction is also problematic: von Fintel (2011) calls it 'linguistically inept'. But getting the linguistic nuances right doesn't, for present purposes, justify the increase in complexity that comes with refining the distinction. So I will continue to use it and reserve the term 'counterfactual' for backward-looking subjunctives.

To explain how and why our readiness to believe and assert conditional sentences, at different times (or informational contexts), varies with mood and tense, the Suppositional theory draws on two key facts: firstly, that there are different ways of supposing something true and, secondly, that different modes of supposition are appropriate for the evaluation of indicative and subjunctive conditionals. We might suppose that as a matter of fact something is true, such as when I suppose, to help with my financial planning, that I won't have enough money at the end of the month to pay the rent. Suppositions of this kind I will call *evidential*, since the proposition supposed true is treated like a new piece of evidence that we have acquired. Evidential supposition should not lead one to give up any beliefs about what is in fact true or to adopt ones one knows to be false. I should not, for instance, adopt the belief that I will secure a large inheritance to cover the rent when reasoning evidentially from the supposition that I will be short at the end of the month.

In contrast to evidential supposition, we might also suppose something to be true, contrary to, or independently of, the known facts, such as when I suppose that it rained yesterday (it did not) in order to consider what I would have done had this been the case. Suppositions of this kind, I will call *interventional*, since we treat the assumed proposition as if it had been made true by an intervention from outside the actual system of causal relationships. Interventional suppositions are often best accommodated by giving up some beliefs that one knows to be true (in actual fact), to allow retention of well-entrenched conceptions about the way that the world works. For example, when supposing that it rained yesterday, in order to think about what I would have done had this been the case, I might have to give up my belief that I went for a walk in the mountains that day, even if I did in fact do so (and have sore feet to prove it).

To harness these distinctions for the purpose of explaining our attitudes to conditionals the Ramsey Test hypothesis must be augmented by postulates about the form of supposition relevant to the evaluation of each kind of conditional. I will make two.

(RT1) An indicative conditional is evaluated *evidentially*, by supposing that its antecedent is, as a matter-of-fact, true. This involves *adding* the antecedent to our current set of beliefs and then determining whether, or to what degree, the truth of the consequent follows.

(RT2) A subjunctive conditional is evaluated *interventionally*, by supposing that, potentially contrary to the facts, its antecedent is true. This involves *accommodating* the truth of the antecedent by suspending our beliefs about its causal preconditions and then inferring the truth or falsity of its consequent, drawing on our beliefs about the causal relationships between the two.

To illustrate and assess these two claims, consider the first pair of (forward-looking) conditionals (1) and (2). Although they make 'opposite' claims, assertion of both is quite reasonable from the point of view of our two RT hypotheses. The first sentence, being an indicative conditional, is evaluated by supposing that Jim will as a matter of fact make the investment. Since he rarely makes a mistake, it is reasonable to infer from the evidence furnished by his action that we were wrong about the bubble and that the investment in ostrich futures will be profitable. On the other hand, although the corresponding subjunctive conditional is also evaluated by supposing that Jim makes the investment, his so-doing is not treated as evidence for the quality of the investment. This is because we accommodate the supposition of his investing by giving up our belief that he is well informed about the state of the market for ostrich futures in order to retain our belief about the causal consequences of investing in the current state of the market. So instead of inferring the state of the market from the fact that Jim has made an investment, we infer that he will lose money from his investment from what we believe about the relationship between the state of the market, investing and securing profits.

Much the same applies to our t_2 attitudes to the second pair of (backward-looking) conditionals. To evaluate indicative (3), we suppose that Jim did in fact invest in ostrich futures by adding the occurrence of him making the investment to our stock of beliefs and inferring from this that the investment will prove to be profitable. To evaluate (4) on the other hand, we suppose, potentially contrary to the facts, that Jim made the investment but without making the adjustment to our beliefs about the quality of the investment that we would have to make if reasoning evidentially. Hence, we infer that his investment would have made a loss.

By t_2 however our informational situation has changed. Consider first the case where we have learnt that (as expected) the ostrich futures have performed badly. No doubt we will be all the more convinced that Jim would not have invested in them. Nonetheless, supposing evidentially that he has in fact done so, he must have lost money. Similarly, unless we think that Jim's investment would have *caused* the ostrich futures to perform well, we must conclude that on the contrary-to-fact supposition that he made the investment, he lost money. This explains why, by t_2, the backward-looking indicative and subjunctive conditionals that we accept coincide in what they assert.

2. Probabilities of Conditionals as Conditional Probabilities

To make precise this informal explanation of the stylised facts represented by our attitudes to the morphologically different conditional sentences concerning Jim's investment, we need to be correspondingly precise about the differences between evidential and interventional supposition and how they should be embedded within the Suppositional Theory. Let us start by stating the Ramsey Test hypothesis in a general form and introducing the formal vocabulary necessary to do so.

Throughout I will assume a background set L of sentences (denoted by italicised capitals) closed under the sentential operations of conjunction $\wedge$, disjunction $\vee$ and negation $\neg$, and subjective probability functions $\Pr_i$ on L that represent the degrees of belief of a rational agent in the L sentences at time t_i (more exactly the degree to which they believe at that time that what these sentences say is true). The sentence $A\wedge B$ will usually be written AB. Sentences expressing the occurrence of an event at a particular time t_i will be denoted by *i*-subscripted capital e.g. X_1 is the sentence asserting that X occurred at t_1. The subscript will be dropped when the time is fixed for, or is irrelevant to, the discussion. Indicative conditionals will be represented by expressions of the form $A_i \rightarrow C$; subjunctive conditionals by expressions of the form $A_i \rightarrowtail C$. Either type is called backward-looking if the time of its assertion is later than t_i and forward-looking otherwise.[3] I will make no assumptions about the logic of these conditionals other than they should obey Modus Ponens.

Most of the literature on the Ramsey Test hypothesis has focused on the version appropriate to indicative conditionals, known as Adams' Thesis, and which asserts that the credibility (and hence assertability) of an indicative conditional is the conditional probability of its consequent given its antecedent. More formally, for any non-conditional sentences $A, C \in L$ such that $\Pr(A) > 0$ and any time t_i:

(Adams' Thesis) $\Pr_i(A \rightarrow C) = \Pr_i(C|A)$

Note that if Adams' Thesis is to apply to backward-looking conditionals even when the (current) probability of its antecedent is zero, then conditional

[3] No implication that the indicative and subjunctive conditionals have different semantic content should be drawn from this choice of formalism. Indeed, I am inclined to believe the contrary: that they have the same content but that this content is evaluated differently depending on mood of the sentence asserting it. But nothing will depend here on whether this is true or not.

probabilities must be defined for zero-probability sentences. For this the usual ratio definition of conditional probability will not suffice and so we must draw on one of the alternative treatments, such as that of Renyi 1955, which allow conditional probability to be defined for conditions that have measure zero.

Now, at both t_0 and t_1, the conditional probability that Jim will make, or has made, a packet, given him making the investment, is high because in almost every credible circumstance in which Jim makes an investment, the investment is profitable. But by t_2 I know how the ostrich futures have performed. And in case they have performed badly, the conditional probability of him having made money, given an investment in them, has to be at or near zero. So Adams' Thesis does a good job explaining my acceptance of the indicative conditionals (1) and (3) at t_0 and t_1 and the conditions under which I would accept or reject indicative (3) at t_2. On the other hand, since my attitudes to the subjunctive conditionals (2) and (4) are different to my attitudes to the corresponding indicatives (1) and (3), they cannot also be explained by Adams' Thesis.

Adams himself recognised very early on (in Adams 1970) that backward-looking indicatives and subjunctives were evaluated differently, offering as evidence his (now famous) example of the difference in our attitude to the past indicative "If Oswald didn't kill Kennedy, someone else did", which we regard as almost certainly true (as Adams' Thesis predicts), and the counterfactual "If Oswald hadn't killed Kennedy, someone else would have", which we don't. He nonetheless speculated that some other conditional probability would explain our attitude to the counterfactual.

One idea, explored and then rejected by both him (Adams 1975) and Edgington (2004), is that the degree to which a counterfactual should be believed at the time of its utterance is the conditional credibility of its consequent, given its antecedent, at the earlier time at which the truth of its antecedent was resolved. Let's call this the Prior Conditional Credence view of the probability of conditionals. More formally, if the truth of the antecedent of the conditional $A_0 \rightarrowtail B$ was resolved at t_0, then at time $t_i \geq t_0$:

(ConCred) $\mathrm{Pr}_i(A_0 \rightarrowtail B) = \mathrm{Pr}_0(B|A_0)$

The Prior Conditional Credence theory offers no account of forward-looking conditionals and hence of our attitudes to the contrasting pair (1) and (2), but its prescriptions do fit with some usage of backward-looking subjunctives. Here is Edgington's example. I say "If I leave before noon, I will be on time for my appointment with the doctor". I am distracted and fail to leave. Later I say (regretfully) "If I had left before noon, I would have made my appointment". It also explains the difference in our attitudes to the two Oswald-Kennedy sentences. For while Adams' Thesis accords with our willingness to accept the indicative, ConCred accords with our unwillingness to accept the matching subjunctive, assuming that the subjective probability at the time of Kennedy's assassination of someone other than Oswald attempting it was very low.

On the other hand, the Prior Conditional Credence view doesn't do very well in our running example. For one thing, it doesn't provide any explanation of the difference in the attitude I take to the counterfactual (4) at t_1 and at t_2, since it predicts that its probability equals my t_0 conditional probability of him making a packet, given that he invests in ostrich futures, at *all* times later than t_0. Furthermore, it predicts that the probability of subjunctive (4) at t_1 will agree with that of indicative (1) at t_0. But it does not. While the conditional probability is high of

Jim making money, given that he invests, the probability is low that he would have made a packet had he invested in what we believe to be a bad venture. In fact, the t_1 probability of counterfactual (4) agrees, not with the forward-looking indicative (1), but with that of forward-looking *subjunctive* (2), while its the backward-looking indicative (3) that agrees with the forward-looking indicative (1). The general message is thus not that the counterfactuals align with forward-looking indicatives uttered earlier but that, *so long as no new relevant information is obtained*, both backward-looking subjunctives and backward-looking indicatives align with their corresponding forward-looking subjunctives and indicatives.

When new information *is* obtained, in particular about the truth of the consequent, then this alignment breaks down. Edgington's example continued: I arrive late for my appointment having failed to leave before noon. Before I can make my excuses to the receptionist, he says "I am surprised you made it. I heard that most trains have been cancelled". I now say to myself "Even if I had left before noon, I would not have made my appointment". Similarly, in our running example, what I learn at t_2 breaks the alignment between the backward-looking conditionals and the corresponding earlier forward-looking one. If I learn that ostrich futures have performed badly then my earlier conviction that if Jim did make the investment then he made money is overturned. So, my earlier acceptance of (3) gives way to its rejection. On the other hand, if I learn that they performed well, then it's my earlier conviction that had Jim made the investment he would lose money that is overturned. So, my t_1 acceptance of (4) turns into rejection of it at t_2.

3. Probabilities of Conditionals as Conditional Chances

The Prior Conditional Credence view is clearly inadequate and it has no current defendants. But a similar theory, that draws on objective conditional probabilities rather than subjective ones, is more promising. According to what I will call the Prior Conditional Chance view one should set one's degrees of belief in a subjunctive to what one takes to be the conditional objective probability or conditional chance, at the time of the resolution of the truth of the antecedent of the conditional (or immediately prior to it), of the truth of its consequent given the truth of its antecedent. More formally, let CH_0 be a random variable ranging over a set $\{\pi^j\}$ of probability functions, measuring the t_0-chances. Then, on the Prior Conditional Chance view, at any time t_i your degree of belief in the counterfactual $A_0 \rightarrowtail C$ should go by your t_i expectation of the t_0 conditional chances of C given that A, i.e.:

(ConCh) $\Pr_i(A_0 \rightarrowtail C) = \sum_j \Pr_i(CH_0 = \pi^j) \,.\, \pi^j(C|A_0)$

Different interpretations of objective probability or chance will yield different instances of this view. Perhaps the most prominent in the literature is that of Brian Skyrms (1980, 1981, 1988) who takes the relevant objective probabilities to be prior propensities. But Moss (2013), Williams (2008), Joyce (1999) and Pearl (2000) all endorse versions of it—more on the latter later on. (Here I gloss over the fact that Skyrms himself did not endorse this exact view because he was sufficiently convinced by the triviality results of Lewis and others to accept that conditionals do not have truth values. For this reason, he presented his claim as pertaining to what he called the Basic Assertability Value of counterfactuals rather than to their probabilities of truth.)

Most of the exponents of the Prior Conditional Chance view propose it only as theory of the credibility of counterfactuals, but my more general formulation allows that it applies to forward-looking subjunctives are well (i.e. that t_i be earlier than t_0). In this form it offers a general explanation, in terms of the difference between objective and subjective probability, for our contrasting attitudes to the indicative and corresponding subjunctive conditionals concerning Jim's investment. While acceptance of (1) follows from the fact that Jim's investment is evidence for ostrich futures being a good investment, our acceptance of (2) follows from our belief that the objective probability or chance of ostrich futures performing well is low. Similarly, for our acceptance of the contrasting conditionals (3) and (4).

The view also seems to capture the way in which our attitude to counterfactual (4) changes between t_1 and t_2 in response to the information acquired about the performance of ostrich futures. For definiteness suppose that you give non-zero credence to just three hypotheses concerning the conditional chances at t_1 of making a packet, given an investment in ostrich futures: that they are zero, that they are a half and that they are one. At t_1 your credence will be concentrated on the first of these, or perhaps the first and second. This explains why at t_1 you regard (4) as credible: the expected conditional chance at that time of making a packet given an investment is low in virtue of the high probability that it is zero. If you learn that, as expected, ostrich futures have performed badly, you will have all the more reason to concentrate your belief on the hypothesis that there was no chance of making a packet from an investment in ostrich futures. But if you learn that ostrich futures have performed well, you are likely to shift probability from the hypothesis that the t_1 conditional chance was zero to the hypothesis that it was one. This explains why you now (at t_2) reject (4).

All of this seems to offer confirmation of the Prior Conditional Chance view. But there is a problem. For the view underestimates the strength of my t_2 attitudes to counterfactual (4). When I learn how ostrich futures have performed, I become *certain* of one or the other of these counterfactuals (depending on whether they performed well or badly). But unless I attached no credibility *at all* at t_1 to the hypothesis that the conditional chance of making money given an investment in ostrich futures is 0.5, my t_2 estimation of this conditional chance will fall short of one. This is because whatever information I get about the performance of ostrich futures is perfectly consistent with this hypothesis. More exactly, on this hypothesis neither a good performance nor a bad one is more likely, given an investment, so the observation of its performance is uninformative regarding the truth of the hypothesis.

Explaining my newfound certainty about the truth of these counterfactuals is an instance of a famous old problem for accounts of conditionals: Morgenbesser's Coin.[4] An indeterministic fair coin is to be tossed and you are invited to bet on it landing heads. You demur, the coin is tossed and lands heads. Your interlocutor says "If you had bet, you would have won". As you don't believe that your betting would have influenced the toss, you are forced to agree. But the fact that the coin has landed heads is no evidence that it is not fair. So your estimate of the prior conditional chances of landing heads stays at 0.5. It would seem to follow that on the Prior Conditional Chance view you should not regret your

[4] This example was reported in Slote 1978, who attributed it to Sydney Morgenbesser.

failure to bet: what your interlocutor says is as likely to be false as it is to be true. But since what the interlocutor says seems true, this view must be rejected.

A variant. You in fact decide to bet and in due course win it. Someone mistakenly believes that you did not bet and says “If you had bet, you would have won”. You reply “Yes, that’s true. In fact, I did bet and I did win”. Your agreement here with your interlocutor is in line with the widely held semantic principle (known as Centring) that the truth of $A \wedge C$ implies that $A > C$. But it is inexplicable if ConCh is correct, for what they have said is, on this account, as likely to be false as it is to be true.

The explanatory problem presented by the Morgenbesser coin problem and its variant is rather different to that presented by the Oswald-killing-Kennedy one. Before the coin has been tossed we are inclined to accept neither the forward-looking indicative “If you bet, you will win”, nor the corresponding subjunctive “If you were to bet, you would win”; after the hands landing has been observed, we are inclined to accept both the backward-looking indicative “If you did bet, you won” and the corresponding counterfactual “If you had bet, you would have won”. So here our attitudes to the indicative and corresponding subjunctive conditional is the same at every moment of time and the challenge is to explain why both change over time in the same way.

So troublesome has Morgenbesser’s Coin been for theories of conditionals in general (and not just for those under consideration here) that it is worth considering an error theory for the intuition that drives it. Consider a similar case involving a deterministic coin that lands heads if and only if some set of initial conditions C hold. The observation that the coin has landed heads licenses the inference that C is the case. From which it does follow that had you bet on heads you would have won, since the coin always lands heads when C holds. Now the error theory I have in mind says that we mistakenly believe this to be true in Morgenbesser’s case as well because we treat it as a deterministic case with epistemic uncertainty about the determining conditions, rather than a truly indeterministic one. But we are wrong: the observed actual outcome of an indeterministic process is completely uninformative with regard to the truth of counterfactual claims about what the outcome would have been had it occurred under different conditions (including those causally independent of the process).

The problem with error theories of this kind, as Edgington (2004) points out, is that they license what can only be regarded as wishful or magical thinking. I turn out to have the winning ticket in a lottery with 10,000 tickets. I say “I am glad that I rubbed my rabbit’s foot. For had I not I would have lost”. On the view that the error theory is designed to uphold, this sentence is very probably true. I don’t think that this is an implication that we should accept. But if we don’t, then we must accept that the Prior Conditional Chance view is false.

It is not difficult to identify where things have gone wrong for the Prior Conditional Chance view. In the Morgenbesser’s Coin case and others like it, what we learn about particular outcomes of chancy processes trumps what we know or believe about the prior chances of these outcomes. Edgington (2004) suggests a simple fix: look not to the prior conditional chances of the outcomes but to the prior conditional chances *updated* by any relevant information subsequently received and attach a degree of belief to the corresponding counterfactual equal to your expectation of these updated chances. More formally, let S be a proposition expressing all relevant events occurring between (and including) the occurrence of the antecedent and the occurrence (but not including) the consequent. Then, what we can call the Updated Conditional Chances view says that at any time t_i:

(UpdConCh) $\Pr_i(A_0 \rightarrowtail C) = \sum_j \Pr_i(CH_0 = \pi^j).\pi^j(C|A_0 \wedge S)]$

The crucial question for the Updated Conditional Chances view is what information is relevant, i.e. what event S we should conditionalize on. According to Edgington:

> The objectively correct value to assign to such a counterfactual [that if *A* had been the case, then *C* would have been] is not (or not always) the conditional chance of *C* given *A* at the time of the fork; but the conditional chance, at that time, of *C* given *A&S* where *S* is a conjunction of those facts concerning the time between antecedent and consequent which are (a) causally independent of the antecedent, and (b) affect the chance of the consequent (Edgington 2004: 21).

Edgington's proposal nicely explains why our attitudes to the counterfactual conditionals change over time. To see this, let us continue to assume that the performance of ostrich futures is independent of whether or not Jim makes an investment in them. Then, while at t_1 the expected chance is low of Jim making money, given the prospective investment in ostrich futures, at t_2 the expected updated chances of him making money will be one or zero, depending on how in fact the ostrich futures performed. This explains my t_2-acceptance of (4). And unlike the Prior Conditional Chance view it correctly predicts not just which counterfactuals we would assert at each time, but also how strongly we would believe them.

What about the troublesome Morgenbesser Coin case? Here, presumably, we want to update the chances by the information that the coin was tossed and did in fact land heads, but not the information that I did not bet on how it would land. And, indeed, the conditional chance of it landing heads, given that it was tossed and landed heads, is one. This explains why we accept the counterfactual "If you had bet, you would have won". The problem is that we should also be willing to assert the counterfactual "If the coin had been tossed, it would have landed heads"; since it's truth is the reason why you would have won had you bet. But Edgington's condition (a) does not allow us to update on the coin landing heads since the heads-landing of the coin is not causally independent of it being tossed. So, her proposal doesn't allow us to predict our attitude to this second counterfactual. And, in general, it fails to ensure the high credibility of "If *A* had been the case, *B* would have been" in cases in which *A* and *B* are both true, because it doesn't prescribe updating the conditional chances of *B* given that *A*, by the truth of *B* whenever *B* depends causally on *A*.

It is tempting to conclude that the Updated Conditional Chances view should dispense with restriction (a), for there will often be facts that are *not* causally independent of the antecedent but which nonetheless affect the chance of the consequent in a manner relevant to the evaluation of the counterfactual itself. But Edgington has a good reason for not allowing such information. Suppose that at t_1 Jim invests, not in ostrich futures, but in a housing development which makes him a large amount of money. So, at t_2, Jim has made a packet. This fact should not by itself ensure the falsity of counterfactual (4), for the performance of the housing market is of no relevance to that of ostrich futures. But the prior conditional chance of Jim making a packet, given that he invests in ostrich futures, updated by the fact that he makes a packet (through his investment in housing) is one. So without Edgington's clause (a), the Updated Conditional Chances view will prescribe disbelief in subjunctive (4), even in cases in which ostrich futures turn out to perform badly. (Clause (a) blocks this inference because Jim's making a packet from housing is *not* causally independent of his (not) investing in ostrich futures.)

These considerations do not decisively refute the Updated Conditional Chances view since it is possible that some other specification of what information is relevant will deliver the goods. But instead of pursuing it further I want to make a different suggestion which, I will argue, avoids the difficulties of all the accounts considered thus far. The proposal is a very simple one: that the credibility of a subjunctive conditional, counterfactual or otherwise, is the expected *posterior* objective conditional probability of C given that A, and not the prior one. But to defend it I must first make a detour.

4. Expert Probabilities

We saw that the credibility of 'matching' indicative and subjunctive conditionals can differ because when evaluating the former, but not the latter, we treat the truth of the antecedent as evidence about the state of the (actual) world. To better understand what difference this makes let's look at how conditionals would be evaluated by a perfect Bayesian reasoner (hereafter called Expert) who, at each point in time, is as informed about the state of the world as it is possible to be at that time. By a perfect Bayesian reasoner I mean someone that makes no mistakes in their probabilistic reasoning, draws all the inferences that they should from what they know and none that they should not. By as informed as it is possible to be, I mean that they are apprised of any truths that it is physically possible to learn. This will include most facts about the past, but not *a posteriori* truths about the future. Nor will it include laws of nature or counterfactuals, these being things about which Expert must form beliefs by inference from what they do know.

This characterisation of Expert leaves open difficult questions about what inferences she should draw from what she knows and exactly what it is possible to know at any point in time. But all that is important for present purposes is that the Expert's degrees of belief at any particular time are as good as they can be at that time. They cannot be improved by more information, because none is available. And they cannot be improved by elimination of errors of reasoning, for they make none. It follows that our own degrees of belief will be as good they can be when we have correctly aligned them with those of Expert. In this sense Expert's credences constitute ideal or objective probabilities: not because they are mind or judgment independent, but because they are the probabilities that our credences should aim at.

In the light of these observations, let us consider how Expert would evaluate our four conditionals concerning Jim and his investments. Suppose for the sake of the exercise that Expert is fully apprised at t_0 of the state of the market and that she infers from this a probability for the profitability of various possible investments and for Jim investing in any one of them (drawing also on any accessible facts about Jim's preferences and beliefs). Suppose that by t_1, she has learnt of Jim's decision and by t_2 of how the various assets have performed.

We saw earlier that my willingness to accept both the indicative (1) and its 'contrary' subjunctive (2) stemmed from the fact that at t_0 Jim's investment was evidentially relevant for me to the question of the state of the market, and hence whether ostrich futures would provide good returns, even though it was causally irrelevant to it. In contrast, Jim investing or otherwise in the ostrich futures is completely *uninformative* for Expert at t_0 regarding the state of the market. For at t_0, Expert knows what the state of the market is and so Jim's decision cannot contain information about it that she does not already hold. It follows that, for Expert, the performance of the various possible investments do *not* depend probabilistically on whether Jim invests in them or not. This is of course trivially true when Expert's knowledge of the state of the market suffices for her to predict with certainty how investments will perform. But the independence of the two holds even when it does not.

Suppose, as seems reasonable, that it is necessary and sufficient for Jim to make a packet from ostrich futures that he invests in them and that they perform well. Then it follows from the probabilistic independence of the two that Expert's t_0 degree of belief in Jim making a packet, conditional on him investing in ostrich futures, equals her t_0 unconditional degree of belief in this investment performing well. But given that the performance of this investment is causally independent of Jim's decision, this just the same as her degree of belief in him making a packet on the interventional supposition of him investing. So, by application of RT1 and RT2, Expert's degree of belief in the indicative "If Jim does invest in ostrich futures, then he will make a packet" will equal her degree of belief in the subjunctive "If Jim were to invest in ostrich futures, then he would make a packet", i.e. she will accept indicative (1) iff she *denies* its subjunctive contrary (2).

This observation holds at later times as well. Learning at t_1 whether or not Jim made the investment in ostrich futures will, for the same reason as before, make no difference to Expert's evaluation of the returns on it. And so at t_1 and t_2 she will accept indicative (3) iff she denies its counterfactual contrary (4).[5] On the other hand, Expert's evaluation of the returns to the investment will, of course, be sensitive to any information she gains about how ostrich futures have performed. If she knows at t_2 that they have performed badly she will deny (3) and accept (4); if she knows that they have performed well, it will be just the other way around. But in each case her evaluation of the indicative conditional and the corresponding subjunctive will be the same. And in each case her evaluation of both will be independent of whether or not Jim made the investment at t_0 or the degree to which she believes he did.

We reach the same conclusion by looking at the issue from the other direction. Because Jim's investment decision is evidentially relevant for me to the state of the market and the latter is the determinant of how well the various investments perform, for me the probability of whether Jim would make a packet, were he to make an investment in ostrich futures, is sensitive to whether or not Jim will in fact make the investment. In contrast, for Expert, since Jim's decision is evidentially irrelevant to the state of the world, whether he would make a packet were to he to invest is probabilistically independent of whether he invests. Now such independence of the Expert's degree of belief in the subjunctive conditional from its antecedent implies that $\Pr(A \rightarrowtail B) = \Pr(A \rightarrowtail B|A) = \Pr(B|A)$. But by Modus Ponens, $\Pr(A(A \rightarrowtail B)) = \Pr(AB)$ and hence $\Pr(A \rightarrowtail B|A) = \Pr(B|A)$. So Expert's probability for the counterfactual equals the conditional probability of its

[5] It is true that since she knows whether or not John has made the investment, it is inappropriate, because misleading, for her to utter the indicative. But this is not to say that she does not have an attitude to it.

consequent given the truth of its antecedent, i.e. *Adams' Thesis holds for Expert's degrees of belief in subjunctive conditionals*, as well as for indicative conditionals.

I am now in a position to state my proposal regarding the probabilities of subjunctive conditionals. I claimed earlier that since Expert's beliefs are as good as they can be, we should align our degrees of belief, both conditional and unconditional, with hers. But since we don't know what these are, the best we can do is adopt the degrees of belief that we expect Expert to have. We have now seen that Expert will adopt as her degree of belief in a subjunctive conditional, her conditional degree of belief in its consequent given its antecedent. So we in turn should set our degrees of belief in the conditional to our expectation of the conditional credence of Expert. More formally, let Ex be a random variable taking values from a set of possible probability functions on L measuring the degrees of belief of Expert at time t. Then according to what I will call the Expert Conditional Credence view, the probability of a subjunctive conditional at time t_i is given as follows:

(ExConCred) $\mathrm{Pr}_i(A \rightarrowtail C) = \mathbb{E}_i\big(Ex(C|A)\big) = \sum_j \mathrm{Pr}_i(Ex = \pi^j).\pi^j(C|A)$

Let's test this proposal against the postulated attitudes to the subjunctive conditionals in our running example. If, at t_0, the market is such that returns on an investment in ostrich futures will be positive, Expert's conditional probability for Jim making a loss given that he makes an investment will be high; if it such that returns will be negative, it will be low. Since we believe the latter to be true, ExConCred prescribes that we believe counterfactual (2) to a high degree. The same applies at time t_1: our evidence regarding Expert's conditional probabilities has not changed and so a high degree of belief in counterfactual (4) is required. But by t_2 we know that Expert either knows that Jim will have made a packet, conditional on his having invested in ostrich futures, or that he will have made a loss, either undermining our degree of belief in counterfactual (4) or confirming it. So our proposal correctly predicts the evolution in our attitudes to the subjunctive conditionals.

What about the Morgenbesser's Coin case? Prior to it being tossed, Expert does not know how it will land (because it's an indeterministic process) and so, plausibly, will adopt a degree of belief of one half on a bet on heads winning. Consequently, the Expert Conditional Credence view prescribes degree of belief of one half in the forward-looking subjunctive 'if you were to bet on heads, you would win' (just what Adams' Thesis prescribes for the forward-looking indicative 'If you bet heads, you will win'). Once the coin has landed heads, Expert will update on this information and so will set her conditional degrees of belief in winning, given a bet on heads, to her prior conditional degrees of belief in winning, given a bet on heads and the coin landing heads, which of course equals one. (It matters not that she knows at this point that no such bet has been made.) Consequently, ExConCred prescribes full posterior belief in the subjunctive 'If you had bet on heads, you would have won' (just as the Adams' Thesis prescribes full posterior belief in the corresponding backward-looking indicative 'If you did bet on heads, you won'). So this account gets the Morgenbesser Coin case right as well.

Earlier we saw that the Updated Conditional Chance view proposed by Edgington adequately explained our attitudes to these (Morgenbesser) sentences, but not our acceptance of the subjunctive "If the coin had been tossed it would have landed heads" because the coin landing heads is not independent of it being tossed. In contrast the Expert Conditional Credence view gets our attitude to this sentence right as well. For Expert's posterior conditional probability for the coin landing heads, conditional on it being tossed, is of course one, since at this point in time she knows how the coin has landed. And in general, the proposed view will correctly prescribe full belief to any subjunctive of the form "If *A* had been the case, *B* would have been" in cases in which *A* and *B* are both known to be true.

That the Expert Conditional Credence view correctly handles these cases is strong evidence in its favour. Let us now turn to some challenges to it. A first worry that might arise at this point concerns whether the conditional probabilities for the performance of ostrich futures given Jim's decision to invest in them are defined in case Jim does not in fact make the investment. And if they are, in virtue adoption of a suitable definition of conditional probability, whether these probabilities can be determined given that they concern counterfactual possibilities. These worries are misplaced. As we noted earlier, frameworks for conditional probability are available which allow that they be defined for conditions of probability zero. And the relevant conditional probabilities can, in this example at least, be determined without difficulty, even though they are not implied by the unconditional probabilities. The performance of the investment, being causally independent of Jim's decision and evidentially irrelevant to the state of the market, will be inferred by Expert from the state of the market alone. So her conditional expectation at any time for the performance of the investments, given Jim's decision, will equal her unconditional expectation at that time for their performance. Similarly, in the Morgenbesser Coin case, her conditional probability, at any time, for the bet on heads winning, in the event of it being made, will equal her unconditional probability for the coin landing heads, irrespective of whether the bet is or was made.

A second worry. The consequences of the proposed view for the examples we have been looking at are no different from those of Edgington's Updated Conditional Chances view when clause (a) restricting updates to information about events causally independent of the antecedent is removed. So how does the Expert Conditional Credence view handle the case that we used to show why this clause is required? Suppose, as before, that at t_0 Jim can invest in either ostrich futures or housing, but not both. Suppose also that Jim chooses to invest in housing in t_1, that housing performs well but that ostrich futures do not, and that Jim duly makes a packet at t_2. Intuitively, in view of the poor performance of ostrich futures, we should deem highly improbable the counterfactual 'If Jim had invested in ostrich futures, he would have made a packet'. Now the Expert Conditional Credence view equates the probability of a counterfactual with (the expectation of) Expert's conditional credence in the consequent, given the truth of its antecedent. But by this point in time Expert knows that in fact Jim has made a packet, albeit from his housing investment not from an investment in ostrich futures. Does this not entail that at t_2 Expert believes to degree one that Jim made a packet, conditional on investing in ostrich futures?

It does not. It is true that Expert's t_0 conditional probabilities for Jim making a packet, given that he invests in ostrich futures, updated by the fact that he makes a packet, must equal one. (This, recall, is why clause (a) is required by the Updated Conditional Chances view: to block the updating by the fact that Jim makes a packet.) But by t_2, Expert knows that Jim did *not* invest in ostrich futures and so her t_2 conditional credence for Jim making a packet, given that he invests in ostrich futures, is not required to equal one, despite the fact that she knows that he did make a packet. On the contrary, since Jim's investment decision is evidentially irrelevant for her to its performance and since Jim makes a packet from ostrich futures (or from housing) only if he invests in it and it performs well, her conditional credence at any time for making a packet, conditional on an investment, simply equals her credence in the investment performing well. But since it is known at t_2 that they have performed badly, the latter equals zero. So the counterfactual 'If Jim had invested in ostrich futures, he would have made a packet' too must have probability zero (for us) since we know that Expert knows that ostrich futures have performed badly.

References

Adams, E.W. 1965, "On the Logic of Conditionals", *Inquiry*, 8, 166-97.

Adams, E.W. 1970, "Subjunctive and Indicative Conditionals", *Foundations of Language*, 6, 1, 89-94.

Adams, E.W. 1975, *The Logic of Conditionals: An Application of Probability to Deductive Logic*, Dordrecht: Reidel.

Adams, E.W. 1976, "Prior Probabilities and Counterfactual Conditionals", in Harper and Hooker 1976, 1-21.

Bradley, R. 2017, *Decision Theory with a Human Face*, Cambridge: Cambridge University Press.

Dowe, P. and Noordhof, P. (eds.) 2004, *Cause and Chance: Causation in an Indeterministic World*, London: Routledge.

Edgington, D. 1995, "On Conditionals", *Mind*, 104, 235-329.

Edgington, D. 2004, "Counterfactuals and the Benefit of Hindsight", in Dowe and Noordhof 2004, 12-27.

Edgington, D. 2008, "Counterfactuals", *Proceedings of the Aristotelian Society*, 108, 1, 1-21.

Fetzer, J.H. (ed.) 1988, *Probability and Causality*, Dordrecht: Springer.

Harper, W.L. and Hooker, C.A. (eds.) 1976, *Foundations of Probability Theory, Statistical Inference, and Statistical Theories of Science*, Vol. I, Dordrecht: Reidel.

Harper, W.L., Stalnaker, R. and Pearce, G. (eds.) 1981, *Ifs. Conditionals, Belief, Decision, Chance, and Time*, Dordrecht: Reidel.

Joyce, J.M. 1999, *The Foundations of Causal Decision Theory*, Oxford: Oxford University Press.

Moss, S. 2013, "Subjective Credences and Semantic Humility", *Philosophy and Phenomenological Research*, 87, 2, 251-78.

Pearl, J. 2009, *Causality*, New York: Cambridge University Press.

Ramsey, F.P. 1929, "General Propositions and Causality", in *Philosophical Papers*, ed. by D.H. Mellor, Cambridge: Cambridge University Press, 1990.

Renyi, A. 1955, "On a New Axiomatic Theory of Probability", *Acta Mathematica Hungarica*, 6, 3, 285-335.

Slote, M. 1978, "Time in Counterfactuals", *The Philosophical Review*, 87, 3-27.

Skyrms, B. 1980, *Causal Necessity*, New Haven: Yale University Press.

Skyrms, B. 1981, "The Prior Propensity Account of Subjunctive Conditionals", in Harper et al. 1981, 259-65.

Skyrms, B. 1988, "Conditional Chance", in Fetzer 1988, 161-78.

von Fintel, K. 2011, "Conditionals", in von Heusinger et al. 2011, 1515-38.

von Heusinger, K., Maienborn C. and Portner, P. (eds.) 2011, *Semantics: An International Handbook of Meaning*, Vol. 2, Berlin: de Gruyter Mouton.

Williams, J.R. 2008, "Chances, Counterfactuals, and Similarity", *Philosophy and Phenomenological Research*, 77, 2, 385-420.

One or Two Puzzles about Knowledge, Probability and Conditionals

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Abstract

Rothschild and Spectre (2018b) present a puzzle about knowledge, probability and conditionals. This paper analyzes the puzzle and argues that it is essentially two puzzles in one: a puzzle about knowledge and probability and a puzzle about probability and conditionals. As these two puzzles share a crucial feature, this paper ends with a discussion of the prospects of solving them in a unified way.

Keywords: Conditionals, Probability, Knowledge, Adams thesis, Closure

1. Introduction

The topic of this paper is a puzzle by Rothschild and Spectre (2018b). This puzzle involves premises about knowledge and probability on the one hand and premises about knowledge, conditionals and probability on the other. If in this puzzle one were to stick to the premises about conditionals, one could learn that a certain form of skepticism is true. Conversely, denying this form of skepticism could lead one to deny certain principles about conditionals. In this way, it could seem that deep questions in epistemology show a surprising and hitherto unnoticed connection to deep questions about conditionals. This would be exciting news.

But there are also reasons to be somewhat skeptical. As I shall argue in this note, the assumptions about conditionals already constitute a puzzle on their own. Moreover, the remaining assumptions in epistemology are (an incomplete) part of a well-known difficulty of squaring seemingly intuitive assumptions about knowledge and probability with similarly intuitive assumptions about a closure-condition on knowledge. Thus, to some extent we have two puzzles in one: a puzzle about conditionals as well as a puzzle about knowledge and probability. But there is also an interesting connection: the puzzle about conditionals has an effect for single-premise inferences which can otherwise only be duplicated with multi-premise inferences. Given that the two puzzles seem to be generated in a similar way, one may conjecture that they might have similar solutions. I explore this possibility in a final section.

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2. The Puzzle

Here is the puzzle presented by Rothschild and Spectre 2018b, which uses the basic setting of Dorr, Goodman and Hawthorne 2014.[1] You know that a thousand fair coins got tossed yesterday. As a matter of fact, not all coins came up heads. According to Rothschild and Spectre (2018b: 473f.), the following five assumptions are plausibly made about this case (I quote the principles directly from their paper):

(Anti-skepticism) You know that not all the coins landed heads.

(Knowledge & Probability) If you are in a position to know something then you cannot assign it a probability of one-half or less.

(Independence) You should treat each of the coin flips as probabilistically independent.

(Restricted Adams Thesis) Where A and B are non-conditional statements about coin-flips in the setup, you should assign a conditional statement of the form *If A then B* as its probability the conditional probability of B given A.

(Restricted or-to-if) If you know a statement of the form *A or B* but you do not know that A is true or that B is true, then you are in a position to know that *if not A then B*.

As Rothschild and Spectre (2018b) show, these five assumptions are jointly inconsistent. Their argument goes like this. Take m to be the least number so that one knows that not all coins came up heads. By **(Anti-skepticism)**, there is such a number and it is ≤ 1000. Given the minimal choice of m, one does not know that any of the first $m - 1$ coins came up heads. And given **(Knowledge & Probability)**, one cannot know that m came up heads. But one does know that either one of the first $m - 1$ coins came up heads or m came up heads. So one may apply **(Restricted or-to-if)** to conclude that one knows that if none of the first $m - 1$ coins came up heads, m came up heads. What is the probability of this conditional? By **(Restricted Adams Thesis)**, it is the conditional probability and by **(Independence)**, this conditional probability is $1/2$. But this contradicts **(Knowledge & Probability)**: one would know a conditional which only has a probability of $1/2$.

Let me make a few initial comments about the five assumptions constituting this puzzle. Note that the first three assumptions are general epistemological assumptions which are not specifically concerned with conditionals. Call them the *epistemological assumptions*. The last two assumptions, in contrast, are specifically concerned with conditionals. Call those the *assumptions about conditionals*.

Almost all of these assumptions are individually controversial. The principle **(Anti-skepticism)** grants the possibility of knowledge which is purely based on

[1] For related discussion see Bacon 2014 and Rothschild and Spectre 2018a.

probabilistic grounds. Many would hold that just as one cannot know on probabilistic grounds alone that one's lottery ticket lost, so one cannot know on probabilistic grounds alone that in a sequence of coin tosses not all coins came up heads. The second assumption, **(Knowledge & Probability)**, is comparatively much more plausible. It is widely held that for a proposition to be known, it must be likely in the light of one's evidence. If one's rational credences are supposed to match one's evidential probabilities, then it seems to follow that one cannot (rationally) assign to a proposition one knows a probability of 1/2 or less. Even if there is no such requirement on rational credences, one may argue in favor of **(Knowledge & Probability)** by assuming that knowledge implies rational belief, which in turn may require a credence higher than 1/2. The last epistemological assumption could easily be mistaken to be true in virtue of the description of the case. It is built into this description that we are dealing with fair coins which all have the same chance of landing heads. That is, it is fair to assume that the *objective chances* are all probabilistically independent. But this is not what **(Independence)** says: it says that one's rational credences (or evidential probabilities) are independent. But this requires some kind of bridge principle (see Bacon 2014 for discussion).

The final two assumptions about conditionals are well-known for being problematic, particularly when they are considered jointly. **(Restricted Adams Thesis)** is intuitively very plausible but also very hard to satisfy. That said, there are various ways of satisfying it, either in innovative truth-conditional settings or in a non-truth-conditional framework. Lastly, **(Restricted or-to-if)** is known for lending support to the material analysis, for it effectively states that knowledge of the material conditional (the disjunction *not-A or B*) puts one in a position to know the natural language conditional. But this would suggest that the natural language conditional cannot be stronger than the material conditional. As it is arguably not weaker, the material analysis would be vindicated.

This should suffice as an initial overview of the issues one faces when addressing the puzzle by Rothschild and Spectre (2018b). Let us now turn to some of the details.

3. The Puzzle and Closure

To begin with, I would like to highlight the perhaps obvious fact that the first three assumptions—which are not about conditionals at all—are already in tension with a well entrenched principle about knowledge: multi-premise closure under conjunction introduction.[2]

I shall focus on the following closure condition:

(Closure) If you know statements $A_1, \ldots, A_n$, then you are in a position to know their conjunction $A_1 \wedge \ldots \wedge A_n$.

Note that this does not mean that one's knowledge is actually closed under conjunction. **(Closure)** merely states that conjunctions of what one already knows

[2] This is acknowledged by Rothschild and Spectre (2018a) in the context of a related discussion. Cf. also the exchange between Hawthorne and Lasonen-Aarnio (2009) and Williamson (2009).

are within one's *epistemic reach* (I take the latter notion from Egan 2007, 8). By way of conjoining bits of what one already knows, one can come to know a conjunction. This is extremely plausible: competent application of the rule of conjunction introduction seems to be one of the most basic ways of extending one's knowledge by way of inference.[3]

The tension between **(Closure)** and the epistemological assumptions in the puzzle primarily arises because **Independence** implies

> **(Less-than-1)** The probability you assign to not all the coins landing heads is less than 1.[4]

The implication is not hard to recognize (see Bacon 2014 for further discussion). If the coin tosses are independent, the probability of all the coins landing heads is computed by multiplying the probability of the individual coin tosses. If the probability of not all coins landing heads were 1, the probability of all coins landing heads would be 0. But for a product to be 0, one of the factors would have to be 0. This, however, would mean that one could exclude with certainty that a certain individual coin landed heads, which, we may plausibly assume, one cannot.

The three assumptions **Anti-skepticism**, **Knowledge & Probability** and **Independence** are not in *direct* conflict with **Closure**, for **Anti-skepticism** merely provides a single proposition which is known despite not being assigned probability 1.

Yet the setup of the example can easily be duplicated. Simply think of m such sequences of coin tosses being realized simultaneously but independent of each other. For every such sequence i, this will provide a statement A_i expressing that not all coins reported in this sequence landed heads. We may assume that all statements A_i are true and true for the same reason (they may even show the same pattern of heads and tails). As we are evidentially positioned in exactly the same way vis-à-vis the statements A_i, one may generalize **Anti-skepticism** to:

> **(Multiple Anti-skepticism)** You know $A_1, \ldots, A_m$.

If **Closure** and **Multiple Anti-skepticism** are added to **Knowledge & Probability** and **Independence** from the original puzzle, we face again an inconsistency. With **Closure**, one can infer from **Multiple Anti-skepticism** that one is in a position to know the conjunction $A_1 \wedge \ldots \wedge A_n$. By **Less-than-1**, each A_i should be assigned a probability less than 1. Given that all the coin tosses are causally independent of one another, they should be treated as independent (by **Independence**). Hence, we can choose an m large enough so that the probability of the conjunction $A_1 \wedge \ldots \wedge A_m$ is below 1/2, which contradicts **Knowledge & Probability**. This is because the probability of the conjunction is going to equal $P(A_1) \times \ldots \times P(A_m)$

[3] For ways of fine-tuning a closure condition on knowledge, see Hawthorne 2005.

[4] Rothschild and Spectre (2018b) are aware of this consequence but treat it as another reason to deny that knowledge requires probability 1.

which is going to drop further and further the more terms with a value less than 1 are added.[5]

Neither **Closure** nor **Multiple Anti-skepticism** are uncontested principles about knowledge. My point here is merely that one can set up a similar puzzle by substituting the assumptions about multi-premise closure for the two assumptions about conditionals in the original puzzle proposed by Rothschild and Spectre (2018b). What one gains is, roughly speaking, a puzzle about knowledge and probability only. This gives us some reason to think that the puzzle is not essentially about knowledge of conditionals, but is more concerned with the intricate issues surrounding knowledge of facts which are the result of probabilistic processes.

To further substantiate this diagnosis, let me highlight a crucial property of probabilities of conjunctions:

(Fact 1: Probability & Conjunction)

Certainty Preservation. Always: If $P(A) = 1$ and $P(B) = 1$, then $P(A \wedge B) = 1$.

Probability Drop. Often (for $x < 1$): $P(A) \geq x$ and $P(B) \geq x$, but $P(A \wedge B) < x$.

Probability 1 is preserved under conjunction introduction. If you are certain about the truth of A and B, you can be certain about their conjunction. However, this does not hold for any threshold less than 1. For instance, as in the type of case which currently interests us, if A and B have equal probability x with $x < 1$ and are probabilistically independent, then their conjunction will be below the threshold set by x. So unless one is willing to give up on any fixed positive threshold for knowledge, be it probability 1/2 or less, one runs into a problem with **Closure** if knowledge can be accumulated while being assigned a probability of less than 1.

4. The Puzzle and Conditional Probability

In a key respect, conditional probabilities relate to disjunctions similarly to how unconditional probabilities relate to conjunctions. The following is the relevant fact I have in mind (it plays an important role in the debate about conditionals and probability; see Adams 1975; Adams 1998, Edgington 1986; Edgington 1995):

(Fact 2: Probability & or-to-if)

Certainty Preservation. Always: If $P(A \vee B) = 1$, then $P(B|\neg A) = 1$ as long as the conditional probability is defined.

Probability Drop. Often (for $x < 1$): $P(A \vee B) \geq x$, but $P(B|\neg A) < x$.

The two observations concern the relation between conditional probabilities and probabilities of disjunctions. A disjunction which is certain forces the corresponding conditional probability to be 1, while a disjunction which is uncertain may go with a comparatively low conditional probability.

[5] Effectively, we have reached a problem that is structurally very similar to the lottery paradox. An obvious difference is that the present puzzle is concerned with knowledge rather than rational belief. Another difference is that the starting assumptions, and hence their conjunction, are all true.

Here is a way of illustrating the two properties. Suppose it is quite likely that either the gardener or the cook was responsible for a murder. Assume further that this is so because it is fairly likely that it was the gardener, while it is almost certain that the cook did not do it. However, let's suppose that the disjunction is uncertain because it could also have been the butler. And compared to the possibility of the cook having committed the crime, it is much more likely that the butler did it. How likely is it that the cook did it given that the gardener did not do it? Rather unlikely, for it is much more probable that on the assumption that the gardener did not do it, it was the butler. So, despite the disjunction being probable, the corresponding conditional probability is rather low, illustrating the probability drop reported in **Fact 2**.

Note how the situation changes if we make the disjunction certain. In our example, this means that we eliminate any chance of the butler being the murderer. Once it is certain that it was either the gardener or the cook, it becomes certain that it was the cook given that the gardener did not do it. This illustrates the preservation of certainty reported in **Fact 2**.

The similarity between **Fact 1** and **Fact 2** is obvious. It would therefore not be surprising if assumptions about conditionals, disjunctions and conditional probability could generate a problem similar to that generated in terms of knowledge, probability and closure.

5. The Puzzle and Conditionals

As it stands, **Fact 2** is unconcerned with (indicative) conditionals. However, it becomes immediately relevant once we combine it with the first assumption about conditionals Rothschild and Spectre (2018b) introduce, namely **Restricted Adams' Thesis** stating that the probability of a conditional just is (at least in a certain class of cases) the corresponding conditional probability. What we then find are two corresponding theses: **(a)** if a disjunction is certain, then the conditional is guaranteed to be certain, but **(b)** if a disjunction is merely probable, the probability of the conditional will often be improbable.

Now, Rothschild and Spectre (2018b) put forward a second assumption about conditionals—**Restricted or-to-if**—concerning the knowability of conditionals: knowing a disjunction $A \vee B$ puts one in a position to know the conditional *if not A, B* (provided one does not know either disjunct). This is effectively an assumption about *single-premise closure*: knowledge of a disjunction enables one to knowingly infer the corresponding conditional. Given the probability drop reported in **Fact 2**, one can already identify the outlines of why this leads to trouble. As knowledge is not required to receive probability 1 (as a joint consequence of **Anti-Skepticism** and **Independence**) but is required to respect a certain minimal threshold (as expressed by **Knowledge & Probability**), the probability drop possible for probabilities less than 1 is incompatible with the or-to-if inference being knowledge preserving. Thus, the two assumptions about conditionals, **Restricted Adams' Thesis** and **Restricted or-to-if**, generate a structurally analogous problem to that of closure under conjunction while requiring only a certain instance of single-premise closure, not multi-premise closure. In both cases, the relevant inferences are certainty preserving but not probability preserving. Once knowledge with probability less than 1 is granted but a certain threshold is still maintained, we run into parallel problems. Still, that we can in this way mirror the problems with multi-premise closure in terms of a possible case of single-premise closure is very surprising. How is this even possible?

What makes this surprising is the fact that probability is always preserved under logically valid inferences. If A implies B and $P(A) \geq x$, then $P(B) \geq x$ by the laws of probability. The reason is, very roughly, that if the set of worlds in which B is true includes the set of worlds in which A is true, then B cannot be less probable than A. However, if **Restricted Adams' Thesis** is assumed, the or-to-if inference does not preserve probability (**Fact 2**). As a matter of fact, the probability drop for the or-to-if inference is without any lower bound: to any $x < 1$ and any $\epsilon > 0$, we can find a case with $P(A \vee B) \geq x$ but $P(B|\neg A) \leq \epsilon$ (the murder example above can easily be precisified to fit this structure; the present observation is extensively discussed in Adams 1975; Adams 1998). Such a drop in probability is not even matched by multi-premise closure: although conjunction introduction does not preserve a threshold for probability, the probability of a conjunction always has a lower bound determined by the sum of uncertainties of each conjunct.

Given that valid inferences are probability preserving while the or-to-if inference is not if Adams' Thesis is assumed, a natural conclusion to draw is that the or-to-if inference is simply not valid. Further evidence for this conclusion can be gathered by observing that none of the existing semantics for conditionals which can accommodate a version of Adams' Thesis validates the or-to-if inference (Bacon 2015, McGee 1989, Van Fraassen 1976). As a final point, recall that the or-to-if inference is (equivalent to) the inference from the material conditional to the indicative conditional of natural language. As the material conditional is defined by $\neg A \vee B$, the or-to-if inference gives us *if* $\neg\neg A, B$. Eliminating the double negation allows us then to derive the indicative conditional from the material conditional. But when indicative conditionals are construed along the lines of Adams' Thesis, the point is usually that they appear to be stronger than the material conditional (the material conditional is easily shown not to satisfy Adams' Thesis).

If the or-to-if inference is invalid by the lights of Adams' Thesis, it is of course a question how this can be squared with this inference being knowledge preserving, as **Restricted or-to-if** would have it. To see the problem, take a case which invalidates this inference, that is, a case in which the disjunction is true but the conditional false. If such cases exist, one should be able to gain knowledge about the disjunction without knowledge of the conditional. This possibility would be backed by the following principle:

> **(Knowledge & Inference)** If A does not imply B, then if A can be known at all, A can be known without being in a position to know B.

With this principle in place, **Restricted Adams' Thesis** and **Restricted or-to-if** can be shown to be incompatible. If **Restricted Adams' Thesis** is true, the or-to-if inference is not valid. By **(Knowledge & Inference)**, the disjunction can be known without being in a position to know the indicative conditional. But this contradicts **Restricted or-to-if** (it is easily verified that the side assumptions necessary for these principles to apply—ignorance about the truth of antecedent and consequent, knowability of the antecedent—pose no problem).[6]

The puzzle about indicative conditionals created by **Restricted Adams' Thesis**, **Restricted or-to-if** and **(Knowledge & Inference)** is a real one. It is not easy to reject any of these assumptions. Thus, the assumptions about conditionals in the original puzzle by Rothschild and Spectre (2018b) already form a puzzle on their own as they are inconsistent with an at least *prima facie* harmless principle such as **(Knowledge & Inference)** (more on this principle below, though).

6. An Intermediate Conclusion

Putting all this together, let me try the following diagnosis. The puzzle about knowing conditionals put forward by Rothschild and Spectre (2018b) is actually two puzzles in one. On the one hand, there is the puzzle about knowledge and probability we already face when we aim at a very modest closure constraint. On the other hand, there is the puzzle of how a logically invalid inference (the or-to-if inference) can still be certainty/knowledge preserving (if indeed it has these two features).

It is definitely interesting that we can substitute the second puzzle for the closure constraint in the first puzzle and still get an inconsistency. This is, as we have seen, because the invalidity of the or-to-if inference generates a similar kind of probability drop we otherwise only find with (valid) multi-premise inferences (valid single-premise inferences are always probability preserving).

Yet in my mind this weakens rather than strengthens the puzzle at hand. It is much more puzzling that a modest closure constraint leads to trouble given the three background principles **Anti-skepticism**, **Knowledge & Probability** and **Independence** than that we gain a puzzle by adding two assumptions about conditionals which already form a puzzle on their own.

Nevertheless, given that the two puzzles embedded in the larger puzzle by Rothschild and Spectre (2018b) are structurally fairly similar—both caused by probability drops in inference—one may wonder whether they also have structurally similar solutions.

7. Towards a Solution

It is beyond the scope of this paper to discuss in any detail the deep questions concerning knowledge, probability and conditionals relevant to the above puzzle(s). What I can do, however, is to sketch one particular solution. This solution is the mirror image of a solution I have given to a similar puzzle involving counterfactuals (Schulz 2017, Chap. 4). By way of conclusion, I will briefly consider what this would mean for the general epistemological assumptions about knowledge and probability.

[6] The structure of this problem is known from the debate about conditionals (Edgington 1986; Edgington 1995, Stalnaker 1975; I discuss it in relation to counterfactuals in Schulz 2017, Chap. 4).

It will be helpful to explicitly consider the relation between indicative conditionals and the material analysis. Very roughly, Adams' Thesis implies that the indicative conditionals is not implied by the material conditional, while the or-to-if inference suggests the opposite. So we can put the puzzle in the following form (see Schulz 2017: 104 for a counterfactual version of it):

The Puzzle (indicative version) For most indicative conditionals c:

(P) It is possible that the probability of the corresponding material conditional is high while the probability of c is low.

(K) It is not possible that one knows the corresponding material conditional without being in a position to know c.

The two problematic assumptions (P) and (K) derive from Adams' Thesis and the or-to-if inference respectively. If something like Adams' Thesis holds, then the probability of the material conditional and the probability of the indicative conditional will not align. The probability of the material conditional will often be high, while the probability of the indicative conditional is low (recall the murder example above). As this indicates how Adams' Thesis supports (P), the second assumption, (K), is almost a mere reformulation of **Restricted or-to-if**. As observed earlier, the or-to-inference could be equivalently stated as the inference from the material conditional to the indicative conditional. But then it states that knowing the material conditional puts one into a position to know the corresponding indicative conditional.

Similar to the discussion above, it is easily seen that there is a tension between (P) and (K). In the light of (P), it seems the material conditional cannot imply the indicative conditional. For if it did, how can the probability of the material conditional oftentimes be higher than the probability of the indicative conditional? After all, the laws of probability guarantee that if A implies B, then the probability of B is never lower than the probability of A. On the other hand, (K) suggests that the material conditional must imply the indicative conditional. For if it did not, there are cases in which the material conditional is true while the indicative conditional is false. If one comes to know the material conditional in such a case, one will not be in a position to know the indicative conditional simply because the indicative conditional is false. This is brought out by **Knowledge & Inference**. In sum, the puzzle is that the material seems to imply and not to imply the indicative conditional. Is there a way out?

Of course, one may simply reject one of the horns of the present dilemma. Adams' Thesis is a highly controversial principle, so there are clearly good theoretical reasons to reject it despite its strong intuitive support. Note, however, that although Adams' Thesis gives rise to (P), (P) itself is a much weaker assumption. On almost all accounts which have the indicative conditional stronger than the material conditional, (P) will still be true. Thus, giving up (P) comes very close to defending the material analysis of the indicative conditional. Instead, one could also try to give up (K), perhaps by finding some kind of explanation of why it holds in many circumstances without being true in full generality.

There is one further option which could easily be overlooked. One may also challenge the principle **Knowledge & Inference**. It is fairly clear that this principle is at best a good approximation to a valid principle. To see that it is false as it stands, let p be the proposition 'snow is white' and consider q to be defined as the conjunction of p with 'I exist', a contingent a priori truth (for discussion, see Schulz 2017, 4.3). It is clear that snow being white does not imply that I exist, that is, p does not imply q. But it seems that any time I am in a position to know that snow is white, I am also in a position to know that snow is white and I exist. This is because I am always in a position to know that I exist. Hence, the principle **Knowledge & Inference** fails. Sometimes, a knowable proposition p can be logically weaker than a stronger proposition q, yet any time one is in a position to know p, one is also in a position to know q.

This might provide a means of escape. Could it be that the indicative conditional is logically stronger than the material conditional (as (P) would have it), but any time one is in a position to know that the material conditional is true, one is also in a position to know that the indicative conditional is true (as (K) would have it)? What would the truth conditions of the indicative conditional have to look like for this to be the case?

In Schulz (2017, 9.3) I consider the following semantics for indicative conditionals, modeled after a similar semantics for counterfactuals. The basic idea is to slightly twist Stalnaker's (1975) semantics for the indicative conditional. According to Stalnaker's semantics, an indicative conditional is true at a world if the consequent is true at the closest epistemically possible world at which the antecedent is true (it is vacuously true if the antecedent is false at all epistemically possible worlds). To make this a bit more precise, let w be a salient possible world and C be a set of worlds epistemically possible at w. On Stalnaker's semantics, we would select a closest antecedent-world from C to fix the truth conditions of a conditional at w. But suppose we do not select a closest antecedent-world from C. Suppose instead that any world in C could be selected and that we leave the selection up to chance. The chance of a consequent-world being selected from the antecedent-worlds in C would be given by the probability of a salient probability function. This probability function assigns a conditional probability to the consequent given the antecedent.

By availing ourselves of an operator, the so-called *epsilon-operator*, we may designate an arbitrarily selected world.[7] More precisely, this world is always an antecedent-world among the epistemically possible worlds (if such exist; otherwise the conditional is vacuously true). The chance of an antecedent-world being a consequent-world is the conditional probability of the consequent given the antecedent. An indicative conditional is then said to be true at a world according to this semantics iff the consequent is true at the arbitrarily selected antecedent-world.

A semantics based on arbitrary selection shares with a Stalnakerian semantics the feature that the truth of the conditional is tied to the truth of the consequent at a single antecedent-world. It differs from a Stalnakerian semantics by leaving the selection of this world in a certain sense up to chance.

[7] For more on the epsilon-operator, see Schulz 2017: Chap. 6.1.

There is clearly a lot to say about a semantics of this kind.[8] However, here I would like to focus on its potential to solve the puzzle about indicative conditionals. The crucial thing to note is that thinking about arbitrary antecedent-worlds is very similar to thinking about conditional probabilities. This is no surprise. An arbitrarily selected antecedent-world will have a certain feature depending on how probable it is that a world has that feature given that it is an antecedent-world.

To see this more clearly, consider the example from above: 'If the gardener did not do it, the cook did'. Now consider a world at which the gardener did not do it, which has been arbitrarily selected from the 'the gardener did not do it'-worlds. How likely is it that this world is a world at which the cook did it? The answer is that this likelihood corresponds to the conditional probability of the cook being the murderer given that the gardener did not do it. In this way, the present semantics puts one in a position to secure a version of Adams' Thesis.

What about knowledge of conditionals on this semantics? This turns on the question of when one can know that an arbitrary object has a certain feature. Consider the case of mathematics, where one frequently introduces names for arbitrary numbers. 'Let n be a prime number not greater than 100', we may stipulate.[9] Can we know that n is an odd number? It seems we cannot, for we cannot exclude that $n = 2$. This suggests that a necessary condition on knowledge about arbitrary selection is that we can only know that an arbitrary F is G if all Fs are Gs. This corresponds to the mathematical practice of concluding that all Fs are Gs after having shown than an arbitrary F is G.

If we apply this observation to our semantics of conditionals, we find that in order to know that an arbitrary antecedent-world is a consequent-world, all antecedent-worlds must be consequent-worlds. In others words, if some antecedent-world is not a consequent-world, one cannot know the corresponding conditional. A consequence of this is that knowledge of an indicative conditional requires the conditional probability to be 1.

The present semantics offers a solution to the puzzle about indicative conditionals. The easy part is to show that (P) can be satisfied. Recall that (P) states that oftentimes, the probability of the material conditional is high, while the probability of the indicative conditional is low. As the indicative conditional is stronger than the material conditional on the present semantics, just like Stalnaker's, there is no obstacle to how (P) can be true. If the material conditional is likely in large parts because the antecedent is likely to be false, the indicative conditional can still be unlikely if most of the antecedent-worlds are worlds at which the consequent is false.

The harder part is to show how (K) can be true. How can it be that any time one is in a position to know the material conditional, one is also in a position to know that the indicative conditional is true? This part is harder because the material conditional does not imply the indicative conditional on the present semantics. But it is not too hard either, for knowing that the material truth conditions are satisfied means that among the epistemically possible worlds, any world is ei-

[8] I discuss it primarily in Schulz 2014 and Schulz 2017. See also Andreas 2018, Cross 2019 and Khoo 2020.

[9] For further discussion, see Breckenridge and Magidor 2012.

ther not an antecedent-world or it is a consequent-world. This means that among the antecedent-worlds, all worlds are consequent-worlds. A fortiori, an arbitrary antecedent-world is a consequent-world. Thus, when one reviews the epistemically possible worlds and finds that all worlds are either not antecedent-worlds or worlds at which the consequent is true, one is in a position to see that an arbitrary antecedent-world is a consequent-world. This can explain how knowing the material conditional puts one in a position to know that the indicative conditional is true. The crucial assumption in this argument is that gaining knowledge of the material truth conditions has an effect on the set of epistemically possible worlds, relative to which the truth conditions of the indicative conditional are defined.[10]

By way of conclusion, let us step back and ask: If this is the right solution to the puzzle about indicative conditionals, what does this mean for the larger puzzle introduced by Rothschild and Spectre (2018b)?

The conditional referred to in the puzzle is this:

> If none of the first $(m-1)$ coins came up tails, m came up tails.

Recall that m was chosen to be the minimal number such that one (allegedly) knows that among the m coins at least one came up tails.

This conditional is established by way of an or-to-if inference from the following disjunction:

> Either one of the first $(m-1)$ coins came up tails or m came up tails.

On the present semantics, knowledge of this disjunction puts one in a position to know the indicative conditional, even though the inference itself is not logically valid. For this reason, one would take the following stance on the epistemic status of this disjunction and the corresponding conditional: either they are both known or if the conditional is not known, then the disjunction is not known either. Let the first option be *option 1* and the second option be *option 2*. I shall consider them in turn.

Option 1. On this option, the disjunction is known and the conditional is known on the basis of the disjunction. This option is challenged by the puzzle because Adams' Thesis implies that the conditional's probability is the corresponding conditional probability which is supposed to be 1/2. But the constraint on knowledge and probability excludes knowledge of a proposition whose probability is only 1/2 or less.

If we hold fixed—as the present option requires—that the disjunction is known, it is instructive to see what the present semantics says about the probability of the conditional. If the disjunction is known, then all epistemically possible worlds will be such that either the antecedent of the conditional is false or where it is true, the consequent will be true as well. But this means that an arbitrarily selected antecedent-world is guaranteed to be a consequent-world, for there are no antecedent-worlds which are not also consequent-worlds. This in turn means that on the present semantics, the indicative conditional is guaranteed to be true

[10] This story is essentially the same as the one given by Stalnaker (1975) in his account of why the material conditional *pragmatically but not semantically* implies the indicative conditional.

and will therefore receive probability 1. If this is so, there is no clash with the constraint that a known proposition should have a probability greater than 1/2.

Of course, one thereby denies that the conditional has a probability of 1/2 (bear in mind, however, that one needs to deny this only if one sticks with option 1 which takes the disjunction to be known). But doesn't this clash with Adams' Thesis? Not necessarily. Adams' Thesis just says that the probability of the conditional is the corresponding conditional probability. And this claim is validated on the present semantics. It is just that the conditional probability is taken to be 1 rather than 1/2, because all epistemically accessible antecedent-worlds are consequent-worlds.

Thus, on option 1, one ultimately takes issue with the intermediate consequence of the puzzle which has it that the conditional probability of m coming up tails given none of $(m-1)$ came up tails is 1/2. But isn't this consequence extremely plausible? That the probability of a coin coming up tails is 1/2 given the outcome of a number of independent tosses just seems to be a fact, for which a lot of empirical support could be mounted.

Now, what clearly is a fact is that the *objective chance* in this case is 1/2. This can be granted, even on the present option. However, what one will disagree with is that the kind of probability which makes the constraint on knowledge and probability plausible is objective chance. What one will hold instead is that for a proposition to be known, it's *epistemic probability* should not be 1/2 or less. So, one can grant that the conditional objective chance is low but simultaneously say that the conditional epistemic probability is high.[11]

But is this a plausible move? I think it is if one accepts the assumptions behind option 1. For on option 1, one grants that propositions can be known which have an objective chance of being false. So one is already prepared to accept that at least in one sense one's epistemic probabilities can depart from the (known) objective chances. So if one grants that knowledge that either one of the first $(m-1)$ coins came up tails or m came up tails, it is not too much of a further commitment to say that the epistemic probability of m coming up tails given that none of the first $(m-1)$ coins came up tails is 1.[12]

There is still a second option, though. Given the somewhat unintuitive consequence of a known conditionals with an objective chance of only 1/2, one may start to doubt that the relevant disjunction from which the conditional was inferred is actually known. This brings us to option 2.

Option 2. Recall that the disjunction states that either one of the first $(m-1)$ coins came up heads or m came up heads. That this disjunction is known is a fairly immediate consequence of **Anti-Skepticism**. The latter principle assumed that one can know that not all 1000 coins came up heads. The number m was then chosen to be minimal with the property that one knows (or is in a position to know) that not all m coins came up heads. Thus, knowledge of the disjunction is established without any detour through assumptions about conditionals.

So, basically, on option 2 one would have to challenge **Anti-Skepticism**. Just to get this out of the way, note that the label is a bit of a misnomer. By denying

[11] See Williamson 2009 for a defense of such a stance.
[12] Yet see Bacon 2014 for critical discussion.

Anti-Skepticism, one is not committed to classical skepticism. What one would be committed to is that knowledge cannot be gained on purely statistical or probabilistic grounds. One could still know that one has hands. But one could not know, based on probabilistic information alone, that a coin is not going to come up heads a thousand (or a million) times in a row. Call this position, for want of a better name, *probabilistic skepticism*.

Probabilistic skepticism is a fairly common position. There are more theoretical as well as more intuitive reasons for it. Lottery cases generate a strong intuition that one cannot know that one's ticket is a loser if the only information one has is the probabilistic set-up of the lottery. This invites the question of how the present case is epistemologically different from a lottery case.[13] There is also a worry that **Anti-Skepticism** is incompatible with the idea that what one is a position to know is a function of the evidence one has. It seems that the evidence one has in a case in which "One of the m coins came up tails" is true is the same as in a case in which all the coins came up tails, for in both cases one would possess the same probabilistic information. Thus, going with option 2 seems to be a fairly easy way out.

In conclusion, there are two ways in which one can respond to the puzzle by Rothschild and Spectre (2018b). On both of these options, one can hold on to the two assumptions about conditionals. This means that one has to take issue with at least one of the three remaining epistemological assumptions. On option 2, one would deny **Anti-Skepticism** by holding that purely probabilistic information is never sufficient for knowledge. On option 1, the situation is somewhat more complicated. The crucial assumption is that knowledge is incompatible with an evidential probability of less than 1 (as defended by Williamson 2000, Chap. 10), while it is compatible with an objective/statistical chance of less than 1. Thus, if the constraint labelled **Knowledge & Probability** is understood by invoking a sense of 'probability' which goes with, or is tied to, objective/statistical chance, then one would deny this constraint. In contrast, if this constraint is concerned with evidential probability, or a kind of rational credence which aims at evidential probability, then one can accept this constraint. This is because a much stronger constraint would hold: not only is knowledge incompatible with an evidential probability of 1/2 or less, it is actually incompatible with any probability less than 1. So, on the present evidential interpretation of **Knowledge & Probability**, one would take issue with the fact that a joint consequence of **Anti-Skepticism** and **Independence** is that knowledge can have a probability of less than 1. One may blame **Anti-Skepticism** for this consequence, in which case the present rebuttal effectively converges with option 2. But one may also blame **Independence**: it may be that the coin tosses are statistically independent, but not evidentially. The reason would be that if one knows that at least one coin came up tails, then this assumption gets evidential probability 1. From this it follows that not all coin tosses are evidentially independent, as long as one assumes that none of the individual coins is evidentially certain to come up tails (see p. 198 for the reasoning behind this).

[13] Rothschild and Spectre (2018b) feel that they are not forced to assimilate their case with lottery cases.

In sum, then, one should either be a probabilistic skeptic or require knowledge to have evidential probability 1. As a matter of fact, one may well adopt both of these claims, for they seem to mutually support each other.

References

Adams, E. 1975, *The Logic of Conditionals*, Dordrecht: Reidel.

Adams, E. 1998, *A Primer of Probability Logic*, Stanford: CSLI Publications.

Andreas, H. 2018, "Moritz Schulz, *Counterfactuals and Probability*", *Notre Dame Philosophical Reviews*, URL: https://ndpr.nd.edu/reviews/counterfactuals-and-probability/.

Bacon, A. 2014, "Giving your Knowledge Half a Chance", *Philosophical Studies*, 171, 373-97.

Bacon, A. 2015, "Stalnaker's Thesis in Context", *The Review of Symbolic Logic*, 8, 131-63.

Breckenridge, W. and Magidor, O. 2012, "Arbitrary Reference", *Philosophical Studies*, 158, 377-400.

Cross, C. 2019, "Moritz Schulz, *Counterfactuals and Probability*", *Australasian Journal of Philosophy*, 97, 414-17.

Dorr, C., Goodman, J. and Hawthorne, J. 2014, "Knowing against the Odds", *Philosophical Studies*, 170, 277-87.

Edgington, D. 1986, "Do Conditionals Have Truth Conditions?", *Critica: Revista Hispanoamericana de Filosofía*, 18, 3-30.

Edgington, D. 1995, "On Conditionals", *Mind*, 104, 235-329.

Egan, A. 2007, "Epistemic Modals, Relativism and Assertion", *Philosophical Studies*, 133, 1-22.

Greenough, P. and Pritchard, D. (eds.) 2009, *Williamson on Knowledge*, Oxford: Oxford University Press.

Hawthorne, J. 2005, "The Case for Closure", in Steup, M. and Sosa, E. (eds.), *Contemporary Debates in Epistemology*, Oxford: Blackwell, 22-43.

Hawthorne, J. and Lasonen-Aarnio, M. 2009, "Knowledge and Objective Chance", in Greenough and Pritchard 2009, 92-108.

Jackson, F. (ed.) 1991, *Conditionals*, Oxford: Clarendon Press.

Khoo, J. 2020, "Moritz Schulz, *Counterfactuals and Probability*", *The Philosophical Review*, 129, 489-95.

McGee, V. 1989, "Conditional Probabilities and Compound of Conditionals", *Philosophical Review*, 98, 485-541.

Rothschild, D. and Spectre, L. 2018a, "At the Threshold of Knowledge", *Philosophical Studies*, 175, 449-60.

Rothschild, D. and Spectre, L. 2018b, "A Puzzle about Knowing Conditionals", *Noûs*, 52, 473-78.

Schulz, M. 2014, "Counterfactuals and Arbitrariness", *Mind*, 123, 1021-55.

Schulz, M. 2017, *Counterfactuals and Probability*, Oxford: Oxford University Press.

Stalnaker, R. 1975, "Indicative Conditionals", *Philosophia*, 5, Reprinted in Jackson 1991, 269-86.

Van Fraassen, B. 1976, "Probability of Conditionals", in Harper, W. and Hooker, C. (eds.), *Foundations of Probability Theory, Statistical Inference, and Statistical Theories of Science*, Vol. I, Dordrecht: Reidel, 261-301.

Williamson, T. 2000, *Knowledge and Its Limits*, Oxford: Oxford University Press.

Williamson, T. 2009, "Replies to Critics", in Greenough and Pritchard 2009, 279-384.

Probability, Evidential Support, and the Logic of Conditionals

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Abstract

Once upon a time, some thought that indicative conditionals could be effectively analyzed as material conditionals. Later on, an alternative theoretical construct has prevailed and received wide acceptance, namely, the conditional probability of the consequent given the antecedent. Partly following critical remarks recently appeared in the literature, we suggest that evidential support—rather than conditional probability alone—is key to understand indicative conditionals. There have been motivated concerns that a theory of evidential conditionals (unlike their more traditional counterparts) cannot generate a sufficiently interesting logical system. Here, we will describe results dispelling these worries. Happily, and perhaps surprisingly, appropriate technical variations of Ernst Adams's classical approach allow for the construction of a logic of evidential conditionals with distinctive features, which is also well-behaved and reasonably strong.

Keywords: Conditionals, Probability, Evidential support, Suppositional, Transitivity.

1. Introduction

According to a very influential view, the assessment of a conditional statement amounts to the assessment of the conditional probability of its consequent given its antecedent. Put forward by Adams in the Sixties and Seventies (see Adams 1966, 1975), this idea has spread over the decades in many research areas in philosophical logic and the philosophy of language (see Bennett 2003 and Edgington 2020 for useful overviews), and ended up becoming a matter of substantial consensus in the psychology of reasoning (see Evans and Over 2004, and Oaksford and Chater 2010). In addition, it served as a building block for popular logical systems (e.g., Hawthorne 1996, Leitgeb 2004, Thorn and Schurz 2014).

The analysis in terms conditional probability may seem natural, but it is bound to miss a crucially important feature, namely the presence of a link of relevance between antecedent and consequent in a compelling conditional statement (see, e.g., Douven 2008, Krzyżanowska, Wenmackers, and Douven 2013, Skovgaard-Olsen, Singmann, and Klauer 2016, Spohn 2015). In this paper, we take this critical remark seriously and explore an alternative view. As we will see, the

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notion of evidential support can be represented in a probabilistic framework and exploited to characterize a logic of conditionals—*evidential* conditionals—conveying the idea that the antecedent provides evidence that the consequent holds.

Probabilistic approaches have been popular among authors (such as Edgington 1986, 2007) who firmly reject the idea of truth-conditions for non-material conditionals. Even though our work is consistent with this line of thought, we do not see our arguments here as incompatible with the view that conditionals have truth-conditions. Conversely, that view does not necessarily rob a probabilistic analysis of its legitimacy and theoretical potential, or so we believe (a similar approach is illustrated in different ways by Crupi and Iacona 2020, Égré, Rossi, and Sprenger 2021, and Douven 2016 himself). As a consequence, we take our treatment of conditionals to be essentially neutral with respect to the question whether conditionals have truth-conditions.

The paper is organized as follows. First, we outline a traditional approach to a probabilistic logic of conditionals (2.), following Adams's (1975, 1998). Second, we explain why evidential support should play a critical role in a satisfactory analysis of indicative conditionals and we reconstruct Douven's (2016) attempt to develop this project (3.). We then address some key principles of conditional logic (4.). Here, we argue that the logic of evidential conditionals should differ from both the traditional account and Douven's (2016) earlier attempt, and a logic with the desired features is presented accordingly (5.). We also analyze one specific case more closely (transitivity, 6.), and finally collect some brief concluding remarks (7.).

2. Conditional Probability and Suppositional Conditionals

Adams (1965) famously argued that the material conditional of classical propositional logic is not satisfactory as a model of the logical behavior of indicative conditionals.[1] Informally, Adams started from the idea that an indicative (simple) conditional can be assigned a degree of "assertability" (Adams 1966). The degree of assertability of "if α then β", in turn, is strictly related to the conditional probability $Pr(\beta|\alpha)$, namely, the probability of the consequent *on the supposition* that the antecedent holds. For this reason, we will call *suppositional* this kind of conditionals. For our purposes, a precise rendition of the ensuing probabilistic logic of conditionals can be given as follows.

Syntax. Let $\boldsymbol{P}$ be a propositional language with a (finite) set of sentence letters p, q, r … and the usual connectives, $\sim$, $\wedge$, $\vee$, $\supset$. Formulas in $\boldsymbol{P}$ are called *propositional formulas*, and $\vDash_{PL}$ will denote classical logical consequence in $\boldsymbol{P}$. We then define a language $\boldsymbol{L}_{\Rightarrow}$ including a further conditional symbol $\Rightarrow$:

– if $\alpha \in \boldsymbol{P}$, then $\alpha \in \boldsymbol{L}_{\Rightarrow}$;
– if $\alpha,\beta \in \boldsymbol{P}$, then $\alpha \Rightarrow \beta \in \boldsymbol{L}_{\Rightarrow}$;
– if $\alpha \in \boldsymbol{L}_{\Rightarrow}$, then $\sim\alpha \in \boldsymbol{L}_{\Rightarrow}$.

Note that $\boldsymbol{L}_{\Rightarrow}$ so defined leaves out embeddings and compounds of formulas with $\Rightarrow$, but allows for (iterated) negation of such formulas.

Semantics. For any standard probability function Pr over $\boldsymbol{P}$, we define a valuation function $V_{Pr} : \boldsymbol{L}_{\Rightarrow} \rightarrow [0,1]$ as follows:

[1] This negative conclusion is now popular. Williamson (2020) is a notable recent exception.

– for every $\alpha \in \boldsymbol{P}$, $V_{Pr}(\alpha) = Pr(\alpha)$;
– $V_{Pr}(\alpha \Rightarrow \beta) = Pr(\beta | \alpha)$, with $V_{Pr}(\alpha \Rightarrow \beta) = 1$ in case $Pr(\alpha) = 0$;
– $V_{Pr}(\sim \alpha) = 1 - V_{Pr}(\alpha)$.

V_P can be seen as representing the degree of assertability of sentences, including simple non-material suppositional conditionals of the form $\alpha \Rightarrow \beta$, and their (possibly iterated) negations.

Validity. For $\alpha_1, \ldots, \alpha_n, \beta \in \boldsymbol{L}_{\Rightarrow}$, $\alpha_1, \ldots, \alpha_n \vDash \beta$ if and only if, for any Pr, $\sum_1^n [1 - V_{Pr}(\alpha_i)] \geq 1 - V_{Pr}(\beta)$. In Adams's (1975, 1998) terminology, the *lack* of assertability of α, namely $1 - V_{Pr}(\alpha)$, is labelled *uncertainty*. So, according to the definition above, an argument is valid if and only if the uncertainty of the conclusion cannot exceed the total uncertainty of the premises. More informally, one can say that in a valid argument a high degree of assertability of the premises implies a high degree of assertability of the conclusion (at least when the premises are not too many). Importantly, as Adams (1998: 151) points out, if $\alpha_1, \ldots, \alpha_n, \beta \in \boldsymbol{P}$, then $\alpha_1, \ldots, \alpha_n \vDash_{PL} \beta$ if and only if $\alpha_1, \ldots, \alpha_n \vDash \beta$, so the probabilistic notion of validity recovers all classically valid inferences for the propositional fragment of the language (also see Suppes 1966). Moreover, substitution of (classically) logically equivalents also holds in this logic (Crupi and Iacona 2021).

The logic of the suppositional conditional $\Rightarrow$ is well understood (see, e.g., Leitgeb 2004, Ch. 3, and Crupi and Iacona 2021). In particular, $\Rightarrow$ is known to validate the following ten important principles, where ">" denotes a generic conditional, "T" stands for tautology, and the long arrow "$\Longrightarrow$" indicates valid inference:

1. *Superclassicality* (SC):	If $\alpha \vDash_{PL} \beta$, then $\alpha > \beta$ must hold
2. *Modus Ponens* (MP):	$\alpha > \beta,\ \alpha \Longrightarrow \beta$
3. *Conjunction of Consequents* (CC):	$\alpha > \beta,\ \alpha > \gamma \Longrightarrow \alpha > (\beta \wedge \gamma)$
4. *Disjunction of Antecedents* (DA):	$\alpha > \gamma,\ \beta > \gamma \Longrightarrow (\alpha \vee \beta) > \gamma$
5. *Cautious Monotonicity* (CM):	$\alpha > \beta,\ \alpha > \gamma \Longrightarrow (\alpha \wedge \gamma) > \beta$
6. *Right Weakening* (RW):	If $\beta \vDash_{PL} \gamma$, then $\alpha > \beta \Longrightarrow \alpha > \gamma$
7. *Limited Transitivity* (LT):	$\alpha > \beta,\ (\alpha \wedge \beta) > \gamma \Longrightarrow \alpha > \gamma$
8. *Rational Monotonicity* (RM):	$\alpha > \beta,\ \sim(\alpha > \sim\gamma) \Longrightarrow (\alpha \wedge \gamma) > \beta$
9. *Conjunction Sufficiency* (CS):	$\alpha \wedge \beta \Longrightarrow \alpha > \beta$
10. *Conditional Excluded Middle* (CEM):	$\sim(\alpha > \beta) \Longrightarrow \alpha > \sim\beta$

It is also well-known that the following principles are *not* valid for the suppositional conditional:

11. *Monotonicity* (M):	$\alpha > \gamma \Longrightarrow (\alpha \wedge \beta) > \gamma$
12. *Transitivity* (T):	$\alpha > \beta,\ \beta > \gamma \Longrightarrow \alpha > \gamma$
13. *Contraposition* (C):	$\alpha > \beta \Longrightarrow \sim\beta > \sim\alpha$

3. The Role of Evidential Support

In order to assess the logical profile of $\Rightarrow$ relative to principles (1)-(13), one of course has to keep in mind what $\alpha \Rightarrow \beta$ is meant to represent in the first place, that is, roughly, a statement that the consequent β is credible *on the supposition that* the circumstance described by the antecedent α obtains. Adams's extensive work has rather effectively supported the adequacy of his logic relative to its target explicandum (see Adams 1965, 1998: Ch. 6), and we can take for granted his view here. However, whether or not the assertability of indicative conditionals is fully captured by the theory of suppositional conditionals is a separate matter. According to McGee (1989), for instance, Adams's theory "describes what English speakers assert and accept with unfailing accuracy" (485), but Douven (2008) has put forward a forceful counterargument. One of Douven's key illustrations involved a long series of tosses of a fair coin, and a comparison between two statements:

(*) If there's a head in the first ten tosses, then there will be a head in the first 1.000.000 tosses.

(**) If Barcelona wins the Champions league, then there will be a head in the first 1.000.000 tosses.

The probabilities of the consequent given the antecedent in (*) and (**) are only minutely different (and one can make them converge at will, by just increasing the number of tosses). And yet, Douven (2008) points out, (**) appears vastly less compelling than (*), which then raises a crucial problem.[2]

Douven's proposed solution is that the plausibility of a (simple, indicative) conditional is accounted for by the conditional probability of the consequent given the antecedent, but only on the additional proviso that the antecedent gives *evidential support* to the consequent. Following the standard probabilistic construal of evidential support or incremental confirmation (e.g., Earman 1992, Fitelson 1999, Crupi 2015, 2020), this means that the conditional probability of the consequent given the antecedent must be *higher than* the unconditional probability of the consequent itself. For our purposes, Douven's own theory (developed in Douven 2016) can be specified as follows.

Syntax. Let $\boldsymbol{P}$ be a propositional language as above. We then define a language $\boldsymbol{L}_{\rightarrow}$ including a further conditional symbol $\rightarrow$:

– if $\alpha \in \boldsymbol{P}$, then $\alpha \in \boldsymbol{L}_{\rightarrow}$;
– if $\alpha,\beta \in \boldsymbol{P}$, then $\alpha \rightarrow \beta \in \boldsymbol{L}_{\rightarrow}$;
– if $\alpha,\beta \in \boldsymbol{L}_{\rightarrow}$, then $\alpha \wedge \beta$, $\alpha \vee \beta$, $\alpha \supset \beta \in \boldsymbol{L}_{\rightarrow}$;
– if $\alpha \in \boldsymbol{L}_{\rightarrow}$ then $\sim\alpha \in \boldsymbol{L}_{\rightarrow}$.

Language $\boldsymbol{L}_{\rightarrow}$ so defined allows for formulas with $\rightarrow$ to appear in the scope of all other connectives, but still leaves out embeddings, i.e., formulas with $\rightarrow$ occurring within the scope of $\rightarrow$ itself.

Semantics. For any standard probability function Pr over $\boldsymbol{P}$ and a threshold t ($1/2 \leq t \leq 1$) we define a valuation function $V_{(Pr,t)} : \boldsymbol{L}_{\rightarrow} \rightarrow \{0,1\}$ as follows.

– For every $\alpha \in \boldsymbol{P}$, $V_{Pr,t}(\alpha) = 1$ if and only if $Pr(\alpha) > t$; and $V_{Pr,t}(\alpha) = 0$ otherwise.

[2] One reaction could be to acknowledge that (*) is largely more assertible than (**) while insisting that they are both *true*, thus semantically on a par (Douven 2016: 105-07, discusses a Gricean variant of this reply, and finds it eventually defective). However, there also exist truth-conditional treatments of conditionals making (*) true and (**) false on very plausible assumptions. A major example is the strict conditional view (see, e.g., Lycan 2001, Gillies 2009, and Kratzer 2012), another one is the modal semantics for the evidential conditional presented in Crupi and Iacona (2020). So the relevance of Douven's example is not limited to a non-propositional view of conditionals, and reliance on the example does not presuppose such a view.

– $V_{Pr,t}(\alpha \rightarrow \beta) = 1$ if and only if: (i) $Pr(\beta | \alpha) > t$ and $Pr(\beta | \alpha) > Pr(\beta)$, or (ii) $Pr(\alpha) = 0$, or (iii) $Pr(\beta) = 1$; $V_{Pr,t}(\alpha \rightarrow \beta) = 0$ otherwise.
– $V_{Pr,t}(\alpha \wedge \beta) = 1$ if and only if $V_{Pr,t}(\alpha) = 1$ and $V_{Pr,t}(\beta) = 1$; $V_{Pr,t}(\alpha \wedge \beta) = 0$ otherwise.
– $V_{Pr,t}(\alpha \vee \beta) = 1$ if and only if $V_{Pr,t}(\alpha) = 1$ or $V_{Pr,t}(\beta) = 1$; $V_{Pr,t}(\alpha \vee \beta) = 0$ otherwise.
– $V_{Pr,t}(\sim\alpha) = 1$ if and only if $V_{Pr,t}(\alpha) = 0$; $V_{Pr,t}(\sim\alpha) = 0$ otherwise.

In Douven's (2016) favorite terminology, $V_{Pr,t}(\alpha) = 1$ and $V_{Pr,t}(\alpha) = 0$ are meant to represent that α is acceptable / not acceptable, respectively, relative to probability distribution Pr and threshold t. So this framework embeds a version of so-called Lockean thesis for qualitative rational acceptability.

Validity. For $\alpha_1, \ldots, \alpha_n, \beta \in \boldsymbol{L}_{\rightarrow}$, $\alpha_1, \ldots, \alpha_n \vDash \beta$ if and only if, for any Pr and any t, if $V_{Pr,t}(\alpha_1) = \ldots = V_{Pr,t}(\alpha_n) = 1$ then $V_{Pr,t}(\beta) = 1$, too. According to this definition, an argument is valid if and only if it preserves rational qualitative acceptability. Substitution of (classically) logically equivalents also holds in this logic. Importantly, however, there exist straightforward cases in which $\alpha_1, \ldots, \alpha_n \vDash_{PL} \beta$ for $\alpha_1, \ldots, \alpha_n, \beta \in \boldsymbol{P}$, and yet $\alpha_1, \ldots, \alpha_n \nvDash \beta$, so many classically valid inferences for the propositional fragment of the language are *not* recovered.

4. A Second Look at Logical Principles

Before presenting a discussion of $\Rightarrow$ and $\rightarrow$ and our own alternative proposal, it is important to emphasize our premises and current goals. First, here we will not challenge the idea that Adams's theory captures the logic of the suppositional conditional as a target explicandum. However, and second, we concur with Douven that a non-material indicative conditional conveying evidential support deserves being modeled to account for certain reliable and important intuitions. In fact, the idea of evidential support clashes with some of the principles that are valid for the suppositional conditional. The principles (6) to (10), in particular, turn out to be increasingly questionable in this perspective. Let us discuss them in turn.

Consider Right Weakening first, one of the most entrenched and technically powerful rules of conditional logic (see, e.g., Nute 1980: 52-3). At least since the debate between Hempel (1945) and Carnap (1962), it is clear that evidential support must fail so-called "special consequence condition": take contingent $\alpha,\beta \in \boldsymbol{P}$, probabilistically independent for some Pr, then surely α will support $\alpha \wedge \beta$ but not β—namely, $Pr(\alpha \wedge \beta \mid \alpha) > Pr(\alpha \wedge \beta)$, while $Pr(\beta | \alpha) = Pr(\beta)$—despite the fact that, obviously, $\alpha \wedge \beta \vDash_{PL} \beta$. Such failure of the special consequence condition arguably carries over to conditionals in the evidential sense. For an illustration, consider an uncertain election with five candidates, Anna, Barbara, Robert, Steven, and Ted, in strictly decreasing order of strength. The conditional "if Anna does not win the election (α), then a woman (Anna or Barbara) will win the election (γ)" is odd in terms of evidential support: in fact, finding out that α must make γ less probable than it was otherwise, and thus provides evidence *against* γ, if anything.[3] Given the assumed ranking of the candidates, however, one might

[3] Proof: given that $\sim\gamma \vDash_{PL} \alpha$ and $1 > Pr(\alpha)$, $Pr(\sim\gamma) > 0$, we have (by Bayes's theorem) that $Pr(\sim\gamma | \alpha) > Pr(\sim\gamma)$, and therefore $Pr(\gamma | \alpha) < Pr(\gamma)$.

well consider "if Anna does not win the election (α), then Barbara will win (β)" compelling, for the antecedent might increase the probability of the consequent significantly. So, intuitively, we have a case where "if α then β" is compelling while "if α then γ" is not, even if β entails γ—which is against RW.

Puzzling cases also arise concerning the related principle of Limited Transitivity (LT entails RW, given SC, see Crupi and Iacona 2020). Here is an illustration. Suppose you don't know who won Wimbledon in July 2018, and consider the following:

α = ATP #2 player in early July 2018 (Nadal) didn't win Wimbledon 2018
β = ATP #1 player in early July 2018 (Federer) did win Wimbledon 2018
γ = ATP #3 player in early July 2018 (Zverev) didn't win Wimbledon 2018

Take the instance of LT consisting in the inference from "if α then β" and "if α and β, then γ" to "if α then γ". The first premise, "if α then β", is plausible: for someone who does not know the outcome, getting to know that Nadal didn't win definitely is favorable evidence for Federer having been the winner.[4] The second premise, "if α and β, then γ", is straightforward; given the obvious background assumption that only one player wins Wimbledon, the consequent γ is implied by the antecedent, α and β. And yet the conclusion "if α then γ" is completely unsound in terms of evidential support: if Nadal didn't win, then the likelihood that a competitor such as Zverev *did* win must increase. As small as such increase may be, it definitely implies that α is *not* evidence in favor of γ.[5] In a case like this, assuming that conditionals convey evidential support, LT would license an inference from plausible premises to a very implausible conclusion.

Rational Monotonicity is also a rather popular principle in the literature. Still, we submit that an evidential conditional "if α then β" does not license the conclusion that "if α and γ, then β", even under the additional proviso that "not: if α then not-γ". Adapting a well-known example (Pearl 1988: Ch. 2), suppose a house alarm (α) can be triggered (normally and appropriately) by burglary (β), but also (rarely and accidentally) by an earthquake (γ). Then it makes sense to say that "if the alarm is activated (α), then burglary is happening (β)" in the evidential sense. But it would not be sensible to conclude that "if the alarm is activated (α) and an earthquake occurred (γ), then burglary is happening (β)". Yet the additional premise "not: if α then not-γ" would also be sound in this case, for the alarm is surely not evidence against the occurrence of an earthquake (in fact, it is an indication in favor of that, although a feeble one).

Conjunction Sufficiency, in turn, is a textbook case of a logical principle which should not be valid for a conditional implying evidential support. Two statements α and β can well jointly hold—say, "Mary went to the party last night" and "there was a full moon last night"—in absence of any connection of evidential support between them, so that "if α then β" (or "if β then α", for that matters) is pointless in the evidential sense. For similar reasons, CEM is clearly ruled out. By denying, for instance, the conditional "if Planet Nine exists, then the European Union will collapse within 5 years" for lack of a connection of evidential support between antecedent and consequent one is in no way logically committed

[4] As a matter of fact, the 2018 winner of Wimbledon was Novak Djokovic, then ranked #21.
[5] Here again, $Pr(\gamma | \alpha) < Pr(\gamma)$ because $\sim\gamma \vDash_{PL} \alpha$ (see footnote 1).

to accept "if Planet Nine exists, then the European Union will not collapse within 5 years". Here again, an underlying formal relationship should be noted, because CEM entails CS (given MP, see Crupi and Iacona 2020).

There is plenty of reasons, then, to think that the logic of a non-material conditional conveying the idea of evidential support should depart from the logic of $\Rightarrow$. Douven's theory accounts for the fact that principles (6)-(10) seem unsound as explained above, but at a very high cost: in fact, only one of principles (1)-(13) is validated by the conditional $\rightarrow$, namely SC (see Douven 2016, § 5.2). This is surely too much of a sacrifice. A failure of Modus Ponens, for instance, makes one doubt whether the very name 'conditional' is appropriate (see, e.g., van Fraassen 1976, p. 277). We conclude that, while Douven's project is important, his specific proposal suffers from significant limitations. The question naturally arises, then, if the idea of evidential support can be developed so that a more robust logic ensues.

5. The logic of Evidential Conditionals

Our proposal is to recover the role of evidential support by revamping a much older view, sometimes associated with the ancient Stoic logician Chrysippus (see Sanford 1989, p. 25, and Lenzen 2019, pp. 15-19, for discussion). According to this view, whether a (simple) conditional statement "if α then β" holds has to do with a relationship of incompatibility between the antecedent, α, and the negated consequent, $\sim\beta$. To flesh this out in probabilistic terms, we will need a probabilistic measure of "incompatibility". How should this be defined? First, it should be symmetric, so that the degree of incompatibility of α with $\sim\beta$ equals the degree of incompatibility of $\sim\beta$ with α for any given probability distribution *Pr*. Second, it should be maximal (that is, 1) in case $Pr(\alpha \wedge \sim\beta) = 0$. Third, it should be minimal (that is, 0) in case α and $\sim\beta$ are either probabilistically independent or positively correlated, namely, if $Pr(\alpha \wedge \sim\beta) \geq Pr(\alpha)Pr(\sim\beta)$. The simplest way to meet these constraints, it turns out, is to represent the degree of incompatibility between α and $\sim\beta$ as:

$$1 - \frac{Pr(\alpha \wedge \sim\beta)}{Pr(\alpha)Pr(\sim\beta)}$$

provided that $Pr(\alpha \wedge \sim\beta) \leq Pr(\alpha)Pr(\sim\beta)$, and 0 otherwise. For the limiting cases in which $Pr(\alpha) = 0$ or $Pr(\beta) = 1$, and thus $Pr(\alpha)Pr(\sim\beta) = 0$, the default option is to say that incompatibility is still maximal (i.e., 1) for then again $Pr(\alpha \wedge \sim\beta) = 0$. Our proposal is just to equate the assertability of an *evidential conditional* "if α then β" to the degree of incompatibility between α and $\sim\beta$ thus defined. Appropriate connections with evidential support are thereby promptly recovered. First, as long as $Pr(\beta) < 1$, "if α then β" turns out to be assertable to some degree at least only if assuming α increases the probability of β, that is, only if $Pr(\beta|\alpha) > Pr(\beta)$. Moreover, "if α then β" is maximally assertable in case α makes β certain, that is, when $Pr(\beta) < 1$ and $Pr(\beta|\alpha) = 1$. In fact, our measure of the incompatibility of α and $\sim\beta$ is identical to the measure of Bayesian confirmation as partial entailment of β by α investigated by Crupi and Tentori (2013, 2014). A probabilistic logic of evidential conditionals can now be spelled out accordingly, as follows.

Syntax. Let $\boldsymbol{P}$ be a propositional language as defined above. We then define a language $\boldsymbol{L}_{\rhd}$ including a further conditional symbol $\rhd$:

– if $\alpha \in \boldsymbol{P}$, then $\alpha \in \boldsymbol{L}_{\rhd}$;
– if $\alpha,\beta \in \boldsymbol{P}$, then $\alpha \rhd \beta \in \boldsymbol{L}_{\rhd}$;
– if $\alpha \in \boldsymbol{L}_{\rhd}$, then $\sim\alpha \in \boldsymbol{L}_{\rhd}$.

Language $\boldsymbol{L}_{\rhd}$ so defined leaves out embeddings and compounds of formulas with $\rhd$, but allows for (iterated) negation of such formulas.

Semantics. For any standard probability function Pr over $\boldsymbol{P}$, we define a valuation function $V_{Pr} : \boldsymbol{L}_{\rhd} \to [0,1]$ as follows.

- For every $\alpha \in \boldsymbol{P}$, $V_{Pr}(\alpha) = Pr(\alpha)$.
- $V_{Pr}(\alpha \rhd \beta) = 1 - \frac{Pr(\alpha \wedge \sim\beta)}{Pr(\alpha)Pr(\sim\beta)}$, if $Pr(\alpha \wedge \sim\beta) \leq Pr(\alpha)Pr(\sim\beta)$, with $V_{Pr}(\alpha \rhd \beta) = 1$ in case $Pr(\alpha) = 0$ or $Pr(\beta) = 1$; otherwise $V_{Pr}(\alpha \rhd \beta) = 0$.
- $V_{Pr}(\sim\alpha) = 1 - V_{Pr}(\alpha)$.

V_P is meant to represent the degree of assertability of sentences, including simple non-material evidential conditionals of the form $\alpha \rhd \beta$ (and their possibly iterated negations).

Validity. For $\alpha_1, \ldots, \alpha_n, \beta \in \boldsymbol{L}_{\rhd}$, $\alpha_1, \ldots, \alpha_n \vDash \beta$ if and only if, for any Pr, $\sum_1^n [1 - V_{Pr}(\alpha_i)] \geq 1 - V_{Pr}(\beta)$. Here, just like in Adams's theory as specified above, an argument is valid if and only if the uncertainty (lack of assertability) of the conclusion cannot exceed the total uncertainty of the premises. So once again we can say that in a valid argument a high degree of assertability of the premises implies a high degree of assertability of the conclusion (at least when the premises are not too many). Just like in Adams's theory, moreover, we have both $\alpha_1, \ldots, \alpha_n \vDash_{PL} \beta$ if and only if $\alpha_1, \ldots, \alpha_n \vDash \beta$ (for $\alpha_1, \ldots, \alpha_n, \beta \in \boldsymbol{P}$) and the substitution of (classically) logically equivalents (Crupi and Iacona 2021).

	$\supset$	$\Rightarrow$	$\rightarrow$	$\rhd$
1. Superclassicality	✓	✓	✓	✓
2. Modus Ponens	✓	✓	✗	✓
3. Conjunction of Consequents	✓	✓	✗	✓
4. Disjunction of Antecedents	✓	✓	✗	✓
5. Cautious Monotonicity	✓	✓	✗	✓
6. Right Weakening	✓	✓	✗	✗
7. Limited Transitivity	✓	✓	✗	✗
8. Rational Monotonicity	✓	✓	✗	✗
9. Conjunctive Sufficiency	✓	✓	✗	✗
10. Conditional Excluded Middle	✓	✓	✗	✗
11. Monotonicity	✓	✗	✗	✗
12. Transitivity	✓	✗	✗	✗
13. Contraposition	✓	✗	✗	✓

Table 1: ✓ = valid; ✗ = invalid. The proofs for $\rhd$ are in Crupi and Iacona 2021.

A summary table of the logical profile of our evidential conditional is displayed above, along with the suppositional conditional $\Rightarrow$ and Douven's conditional $\rightarrow$. (The classical material conditional $\supset$ is also included for comparison.)

6. Shades of Transitivity

The pattern of validities / invalidities in Table 1 illustrates that our evidential conditional $\rhd$ overcomes the weakness of Douven's, and it also shows that its logic substantially differs from the logic of the suppositional conditional. A key feature in this respect has to do with Contraposition, which is validated by $\rhd$ but not by $\Rightarrow$. Originally, Adams argued that Contraposition should fail for indicative conditionals (e.g., Adams 1975, pp. 14-15, and Adams 1998, § 6.3). However, the clearest and strongest counterexamples against Contraposition imply that $\alpha > \beta$ can convey an "even if α, β" (or "if α, still β") construction in natural language (see Lycan 2001, p. 34, and Bennett 2003, pp. 32, and 143-144; also see Gomes 2019). This means that such counterexamples become innocent for a non-material conditional connective which clearly *rules out* "even if α, β" as a target explicandum. And that, we submit, is just the case for $\rhd$ (see Douven 2016, p. 119, for an important discussion along the same lines).

Other principles, as we know, are validated by the suppositional conditional but not by the evidential conditional. Our discussion of such cases is not complete yet. Above, we have provided intuitive motivations why principles (6)-(10) are naturally seen as invalid if $\alpha > \beta$ is meant to capture a relation of evidential support from α to β. One could, however, retain a general worry about the failure of Right Weakening and Limited Transitivity. Concerning the suppositional conditional, what such principles tell us is that, while full transitivity fails, there are weaker forms of transitivity that survive in the logic. In fact, the validity of both RW and LT turn out to follow from the validity of Transitivity under unproblematic assumptions. Concerning RW, posit $\beta \vDash_{PL} \gamma$ and assume $\alpha > \beta$. Given Superclassicality, we then have $\beta > \gamma$, and if Transitivity holds we immediately conclude that $\alpha > \gamma$. As for LT, we assume $\alpha > \beta$ and $(\alpha \wedge \beta) > \gamma$, since $\alpha > \alpha$ by SC, we derive $\alpha > (\alpha \wedge \beta)$ from the first assumption by Conjunction of Consequents, and again we conclude that $\alpha > \gamma$ by Transitivity. In the case of Monotonicity and Cautious Monotonicity, the attractive feature of preserving a weaker but discernable variant of a more traditional logical principle is shared by $\Rightarrow$ and $\rhd$. But given that $\rhd$ fails both RW and LT, a natural question is whether there exist *any* weak but non-trivial form of transitivity which is preserved in the logic of evidential conditionals.

Interestingly, at least two of them exist. The first one is appropriately seen as a weakening of Right Weakening itself:

Weak Right Weakening (WRW): If $\beta \vDash_{PL} \gamma$, then $\alpha > \beta$, $\gamma > \beta \vDash \alpha > \gamma$.

As a form of weakened transitivity, this inference rule demands something more from the second link of the transitive chain: not only has γ to follow logically from β (as in plain RW), one must also have the "reverse" evidential conditional $\gamma > \beta$ as a separate premise. As shown by the derivation below, WRW holds given CM, C, and Substitution of Logical Equivalents (SLE), all of which are valid principles for the evidential conditional (Crupi and Iacona 2020).

Assume $\beta \vDash_{PL} \gamma$.

1	$\alpha > \beta$	
2	$\gamma > \beta$	
3	$\sim\beta > \sim\alpha$	[1, C]
4	$\sim\beta > \sim\gamma$	[2, C]

5 $(\sim\beta \wedge \sim\gamma) > \sim\alpha$ [3,4, CM]
6 $\sim\sim\alpha > \sim(\sim\beta \wedge \sim\gamma)$ [5, C]
7 $\alpha > (\beta \vee \gamma)$ [6, SLE]
8 $\alpha > \gamma$ [7, SLE, because $\beta \vDash_{PL} \gamma$]

A symmetric maneuver generates another inference rule:[6]

Weak Left Strengthening (WLS): If $\alpha \vDash_{PL} \beta$, then $\beta > \alpha$, $\beta > \gamma \vDash \alpha > \gamma$.

In this case, the *first* link of the transitive chain is the target of a stringent demand: β has to follow logically from α, and the "reverse" evidential conditional $\beta > \alpha$ must be in place, too. As shown by the derivation below, rule WLS holds given CM and SLE.

Assume $\alpha \vDash_{PL} \beta$.
1 $\beta > \gamma$
2 $\beta > \alpha$
3 $(\beta \wedge \alpha) > \gamma$ [1,2, CM]
4 $\alpha > \gamma$ [3, SLE, because $\alpha \vDash_{PL} \beta$]

Interestingly, one can also easily show that WRW and WLS are interderivable given C. We illustrate below by the right-to-left derivation.

Assume $\beta \vDash_{PL} \gamma$
1 $\alpha > \beta$
2 $\gamma > \beta$
3 $\sim\beta > \sim\alpha$ [1, C]
4 $\sim\beta > \sim\gamma$ [2, C]
5 $\sim\gamma > \sim\alpha$ [3, 4, WLS, because $\sim\gamma \vDash_{PL} \sim\beta$]
6 $\alpha > \gamma$ [5, C]

So in the logic of the evidential conditional, weakened forms of transitivity require that the first (WLS) or second (WRW) link in the chain be strengthened in a similar way. This neat symmetry is broken, instead, in the logic of ⇒: due to RW, strengthening the second link with propositional logical entailment suffice to preserve validity; due to the failure of Monotonicity, however, strengthening the first link with propositional logical entailment does *not* preserve validity.

7. Conclusion

A suppositional view understands a conditional as a statement that the consequent is credible given the antecedent. In many cases, however, a stronger connection between antecedent and consequent seems to be required, one that goes beyond a high conditional probability of the latter given the former. In fact, a number of suggestions broadly along these lines have flourished in recent times, spanning a variety of approaches (Andreas and Günther 2019, Berto and Özgün 2021, Crupi and Iacona 2020, 2021, Raidl 2021, van Rooij and Schulz 2019, Rott 2019). Here, we have suggested a new route to a logic of evidential conditionals, which hinges on the idea that the evidential support from α to β amounts to the

[6] Given SC, WLS can also be seen as a weakened form of another popular rule known as Conditional Equivalence: $\alpha > \beta$, $\beta > \alpha$, $\beta > \gamma \vDash \alpha > \gamma$ (see Adams 1998: 156, Krauss, Lehmann, and Magidor 1990: 179).

degree of incompatibility between α and $\sim\beta$, which also complies with earlier work in Bayesian confirmation theory (Crupi and Tentori 2013, 2014). We have then attached degrees of evidential support to non-material indicative sentences in a suitable formal language, and otherwise retained Adams's definition of validity unscathed. The features of the resulting logical theory are attractive enough, we submit, to motivate serious consideration.

References

Adams, E.W. 1965, "The Logic of Conditionals", *Inquiry*, 8, 166-97.

Adams, E.W. 1966, "Probability and The Logic of Conditionals", in Suppes, P. and Hintikka, J. (eds.), *Aspects of Inductive Logic*, Amsterdam: North Holland, 265-316.

Adams, E.W. 1975, *The Logic of Conditionals*, Dordrecht: Reidel.

Adams, E.W. 1998, *A Primer of Probability Logic*, Stanford: CSLI Publications.

Bennett, J. 2003, *A Philosophical Guide to Conditionals*, Oxford: Clarendon Press.

Berto, F. and Özgün, A. 2021, "Indicative Conditionals: Probabilities and Relevance", *Philosophical Studies*, https://doi.org/10.1007/s11098-021-01622-3.

Carnap, R. 1962, *Logical Foundations of Probability*, second edition, Chicago: University of Chicago Press.

Crupi. V. 2015, "Inductive Logic", *Journal of Philosophical Logic*, 44, 641-50.

Crupi, V. 2020, "Confirmation", in Zalta, E.N. (ed.), *Stanford Encyclopedia of Philosophy*, URL = https://Plato.Stanford.Edu/Archives/Spr2020/Entries/Confirmation/

Crupi, V. and Iacona, A. 2020, "The Evidential Conditional", *Erkenntnis*, https://doi.org/10.1007/s10670-020-00332-2.

Crupi, V. and Iacona, A. 2021, "Three Ways of Being Non-Material", *Studia Logica*, https://doi.org/10.1007/s11225-021-09949-y.

Crupi, V. and Tentori, K. 2013, "Confirmation as Partial Entailment: A Representation Theorem in Inductive Logic", *Journal of Applied Logic*, 11, 364-72 [Erratum In *Journal of Applied Logic*, 12, 2014, 230-31].

Crupi, V. and Tentori, K. 2014, "Measuring Information and Confirmation", *Studies in the History and Philosophy of Science*, 47, 81-90.

Douven, I. 2008, "The Evidential Support Theory of Conditionals", *Synthese*, 164, 19-44.

Douven, I. 2016, *The Epistemology of Conditionals*, Cambridge: Cambridge University Press.

Earman, J. 1992, *Bayes or Bust?*, Cambridge, MA: The MIT Press.

Edgington, D. 1986, "Do Conditionals Have Truth Conditions?", *Critica*, 18, 3-30.

Edgington, D. 2007, "On Conditionals", in Gabbay, D.M. and Günthner, F. (eds.), *Handbook Of Philosophical Logic,* Vol. 14, Berlin: Springer, 127-221.

Edgington, D. 2020, "Indicative Conditionals", in Zalta, E.N. (ed.), *Stanford Encyclopedia Of Philosophy*, URL = https://Plato.Stanford.Edu/Archives/Win2014/Entries/Conditionals/

Égré, P., Rossi, L. and Sprenger, J. 2021, "De Finettian Logics of Indicative Conditionals. Part I: Trivalent Semantics and Validity", *Journal of Philosophical Logic*, 50, 187-213.

Fitelson, B. 1999, "The Plurality of Bayesian Measures of Confirmation and the Problem of Measure Sensitivity", *Philosophy of Science*, 66, 362-78.

Gillies, A.S. 2009, "On Thuth-Conditions for *If* (but Not Quite Only *If*)", *Philosophical Review*, 118, 325-49.

Gomes, G. 2019, "Meaning-Preserving Contraposition of Conditionals", *Journal of Pragmatics*, 152, 46-60.

Hawthorne, J. 1996, "On the Logic of Non-Monotonic Conditionals and Conditional Probabilities", *Journal of Philosophical Logic*, 25, 185-218.

Hempel, C.G. 1945, "Studies in the Logic of Confirmation", *Mind*, 54, 1-26.

Krauss, S., Lehmann, D. and Magidor, M. 1990, "Non-Monotonic Reasoning, Preferential Models, and Cumulative Logics", *Artificial Intelligence*, 44, 167-207.

Kratzer, A. 2012, *Modals and Conditionals*, Oxford: Oxford University Press.

Krzyżanowska, K., Wenmackers, S. and Douven, I. 2013, "Inferential Conditionals and Evidentiality", *Journal of Logic, Language, and Information*, 22, 315-34.

Leitgeb, H. 2004, *Inference on the Low Level*, Dordrecht: Kluwer.

Lenzen, W. 2019, "A Fresh Look at the Stoics' Debate on the Nature of Conditionals", Manuscript.

Lycan, W.G. 2001, *Real Conditionals,* Oxford: Oxford University Press.

McGee, V. 1989, "Conditional Probabilities and Compounds of Conditionals", *Philosophical Review*, 98, 485-541.

Nute, D. 1980, *Topics in Conditional Logic*, Dordrecht: Reidel.

Oaksford, M. and Chater, N. (eds.) 2010, *Cognition and Conditionals: Probability and Logic in Human Thought*, Oxford: Oxford University Press.

Pearl, J. 1988, *Probabilistic Reasoning in Intelligent Systems*, San Mateo (California): Kauffmann.

Raidl, E. 2021, "Definable Conditionals", *Topoi*, 40, 87-105.

Rott, H. 2019, "Difference-Making Conditionals and the Relevant Ramsey Test", *Review of Symbolic Logic*, DOI: 10.1017/S1755020319000674.

Sanford, D. 1989, *If P, Then Q*, London-New York: Routledge.

Skovgaard-Olsen, N., Singmann, H. and Klauer, K.C. 2016, "The Relevance Effect and Conditionals", *Cognition*, 150, 26-36.

Spohn, W. 2015, "Conditionals: A Unifying Ranking-Theoretic Perspective", *Philosopher's Imprint*, 15, 1-30.

Suppes, P. 1966, "Probabilistic Inference and the Concept of Total Evidence", in Suppes, P. and Hintikka, J. (eds.), *Aspects of Inductive Logic*, Amsterdam: North Holland, 49-65.

Thorn, P.D. and Schurz, G. 2014, "A Utility-Based Evaluation of Logico-Probabilistic Systems", *Studia Logica*, 102, 867-90.

Van Fraassen, B.C. 1976, "Probabilities of Conditionals", in Harper, W.L. and Hooker, C.A. (eds.), *Foundations of Probability Theory*, *Statistical Inference, and Statistical Theories of Science*, Vol. I, Dordrecht: Reidel, 261-301.

Van Rooij, R. and Schulz, K. 2019, "Conditionals, Causality, and Conditional Probability", *Journal of Logic, Language, and Information*, 28, 55-71.

Williamson, T. 2020, *Suppose and Tell*, Oxford: Oxford University Press.

Towards a Unified Theory for Conditional Sentences

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Abstract

A unified shared theory of conditionals does not still exist. Some theories seem suitable only for indicative but not for counterfactual ones (or vice versa), while others work well with simple conditionals but not compound ones. Ernest Adams' approach—one of the most successful theories as far as indicative conditional are concerned—is based on a reformulation of *Ramsey's Test* in a probabilistic thesis known as "The Equation". While the so-called *Lewis' Triviality Results* support Adams' view that conditionals do not express genuine statements, the problem arises whether these results lead inevitably to Adams' view—according to which conditionals always lack truth-values—or to the less radical view by Dorothy Edgington—according to whom *simple* (indicative) conditionals have well-defined truth-values only when they are used to make assertions and their antecedent is true.

I will suggest that Alberto Mura's account—a refinement of de Finetti's theory of tri-events that fits Adams' logic and extends it over the lattice of compound conditionals—can be a suitable candidate for a proper semantics of indicative conditionals and might be an interesting step towards a unified theory for conditional sentences.

Keywords: Indicative conditionals, Compound conditionals, de Finetti's tri-events, Adams' logic of conditionals, Lewis' Triviality Results.

1. Introducing Conditional Sentences

Conditional statements have been the subject of several discussions since ancient age. Indeed, linguistic constructions like "If *p*, (then) *q*" have always interested many philosophers and logicians because of their central role in common reasoning: every day we think and act under conditional statements. Unfortunately, the use of these sentences in theoretical and practical reasoning is quite tricky, often leading to absurdities:

[1] "If Trump dies, Biden will win the Elections. If Biden wins the Elections, Trump will resign immediately after the Elections. Therefore, if Trump dies, Trump will resign immediately after the Elections".

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[2] "If Tom were not at home, the lights would be off. The lights are on. Therefore, Tom must be at home".

Argument [1] represents a classical transitive schema whose premises may be both plausible while its conclusion is surely false, although it is a valid instance of a deductively valid argument. Argument [2] is a typical non-inclusive theoretical reasoning that might be easily invalidated by the additional information that sometimes Tom forgets to switch off home lights.

No less problematic is the use of conditional statements in practical reasoning:

[3] "I have heart disease. If I take medicines, I decrease the odds of a heart attack. So, I should take medicines".

Looking at example [3] from another perspective, it seems that people taking those medicines could have a heart attack easier than others. Misunderstanding like this could mislead the decision-maker! So, it is very important to pay attention to the action every conditional is affecting.

In common language, we deal with different types of conditional statements: simples, compounds, indicatives and counterfactuals. We call simple conditionals those sentences of the form $p \rightarrow q$ where "$\rightarrow$"[1] does not occur neither in p nor in q and compound conditionals those compound sentences containing occurrences of conditional connectives in some of its proper sub-sentences. All sentences above are simple conditionals and also those conditionals whose antecedent or consequent contains a connective other than "$\rightarrow$" (for instance '&', '$\vee$', '$\supset$', '$\sim$') belong to this category.

[4] "If Robert buys the eggs and does not break them, then I will make a cake and take it to my mother" [p & $q \rightarrow z$ & v]

[5] "If I win the lottery or inherit a fortune, then I will buy either a villa in Sardinia or an apartment in New York" [$p \vee q \rightarrow z \vee v$]

[6] "If it is a beautiful day, then if I find a ride, I will go to the beach" [$p \rightarrow (\mathrm{q} \rightarrow z)$]

Only sentence [6] is a compound conditional, while [4] and [5] are simple conditionals.

Regarding the difference between indicative and counterfactual conditionals, probably there is no better way than the following example to understand it:

[7] "If Oswald did not shoot Kennedy, someone else did" (*Non-counterfactual or indicative conditional*)

[8] "If Oswald had not shot Kennedy, someone else would have" (*Counterfactual or subjunctive conditional*)

This situation is a paradigmatic illustration because the first proposition is unquestioned, and the second is typically denied.[2] Indeed, unless we are not any conspiracy theorist, we can reject [8] despite accepting [7]. Another dissimilarity—although not crucial—is that [8] presents a modal aspect, namely a *necessary link* (logical or causal) between the antecedent and the consequent, which seems to be missing in [7]. So, the distinction between indicative and counterfactual

[1] Generally conditional connective is represented by '$\rightarrow$', but we can find other symbols too ('$\supset$', '>', etc.), according to different interpretations. I assume that the conditional symbol '$\rightarrow$' is not a truth-function, and in particular that it is different from the material conditional '$\supset$'.

[2] Adams 1970: 89-94.

conditionals is unquestionably pointed out by this example, at the expense of those aspiring to a unified theory simply denying this difference.[3]

We must clarify why counterfactual conditionals are usually identified with subjunctives and non-counterfactuals with indicatives, although there is no complete coincidence.

First, we say that a conditional statement is a *counterfactual conditional* when its antecedent is false.

Second, we say that a conditional statement is a *subjunctive conditional* when the English grammar requires 'would' in the main clause and past tense in the if-clause. It can happen that sometimes these different properties—interpretative and morphological—do not coexist at all so that some subjunctive conditionals do not exclude the possibility of a true antecedent:

[9] "If Chris went to the party this evening, and she probably will go, Tom would be enthusiastic.

In the same way, it may be possible to use indicative conditionals even if we know the antecedent is false:

[10] "If he is handsome, then I am Naomi Campbell!"

However, many philosophers hold it would be wrong to describe a counterfactual merely as a conditional whose antecedent is false. Rather, it would be better to identify it as a proposition that *invokes* in some way the antecedent's falsity.[4] Indeed when we say:

[11] "If Jones were present at the meeting, he would vote for the motion"

instead of:

[12] "If Jones is present at the meeting, he will vote for the motion"

we are pointing out a piece of information rather than another one: with [11] the speaker wants to focus the attention on what Jones would do if he were present at the meeting—without excluding the fact that he could *not* be present (so *invoking the antecedent's falsity*)—instead of [12] it is not important that part of the content about Jones' presence (or absence) but, rather, the information concerning the fact he intends to vote for the motion.

Let me present another example:

[13] "If I went to the prom, would you come with me?"

[14] "If I go to the prom, will you come with me?"

In front of these two utterances, the first thought is that saying [13] is trying to invite me to the prom—he says he would like to go to the prom with me. Instead, about [14] I could think that the guy (maybe a neighbour) is offering me just a

[3] "Therefore, there really are two different sorts of conditional; not a single conditional that can appear as indicative or as counterfactual depending on the speaker's opinion about the truth of the antecedent" (Lewis 1973: 3).

[4] "It is not their [the antecedent's and consequent's] falsity in fact that puts a 'counterfactual' conditional into this special class, but the user's expressing in the form of words he uses, his belief that the antecedent is false" (Mackie 1973: 71).

"'Counterfactual' may seem to be less open to objection. What lies behind this piece of terminology is not, of course, that the antecedent is in fact false, but that, in some way, the falsehood of the antecedent is implied, whether the conditional is true or false, well supported or not" (Woods, Wiggins and Edgington 1997: 5).

ride to the party (maybe by car)—I should be self-confident to think this utterance means a romantic date.

In other words, with [11] and [13] we want to remark just that necessary link between the antecedent and the consequent characterizes, as formerly said, the counterfactual conditionals rather than the indicative ones. Therefore, if we do not strictly denote counterfactuals with those conditionals whose antecedent is false, [11] and [13] could be easily considered counterfactuals as much as the following conditionals:

[15] "If Jones had been present at the meeting, he would have voted for the motion".

[16] "If I had gone to the prom, would you have come with me?"

Furthermore, if we only accepted sentences like [15] and [16] as counterfactuals and rejected [11] and [13], we should consequently treat the last ones such as contrary-to-facts. In this way, we would end to confuse the two well-defined classes. One must establish their differences, and a conditional does not have to work as a supporter for the other one.[5] Examples [7] and [8] entirely show this idea by examples: people who accept [7] hardly hold [8]. Instead, a person could easily accept both [11] and [15]—such as [13] and [16]—recognizing in them the same counterfactual conditional in two different times.[6] So, in order to facilitate, many philosophers—and I agree—have decided to deal with, in general, subjunctive conditionals as counterfactuals and indicative conditionals as non-counterfactuals.

In short, there are different types of conditional statements and to deal with all of them is not problem-free. It is not exhaustive to identify "If p, (then) q" simply with a sentence characterized by a link between an antecedent (p) and a consequent (q). A theory of conditionals must show the great importance of conditional statements when they are acceptable and true or simply assertive. For certain, this is not an easy task, and, although this field has made much progress, a genuinely unified theory of conditionals does not exist yet. Indeed, some theses seem good only for indicative and not for counterfactual conditionals (or vice versa) while others work well with simple conditionals but not with compound ones. A so-called unified theory should apply to all of these different accounts of conditionals.

2. The Equation and Adams' Thesis

Let us consider the famous remark—and footnote related—in Ramsey 1929 and some different suppositional theories born as their interpretation:

> Now suppose a man is in such a situation. For instance, suppose that he has a cake and decides not to eat it because he thinks it will upset him, and suppose that we consider his conduct and decide that he is mistaken. Now the belief on which the man acts is that if he eats the cake he will be ill, taken according to our above

[5] "As has been recognized, what would count as strong, or conclusive, support for a non-counterfactual conditional would not support the corresponding counterfactual" (Woods, Wiggins and Edgington 1997: 7).

[6] Surely, an indicative conditional could become counterfactual with the time, but this is not a proper distinguishing feature, such as examples [4] and [5] shows—neither [4] correspond to [5] or it is [5] in a second moment. At most the indicative "If Oswald didn't kill Kennedy, someone else did" could correspond to some kind of counterfactual like "If Oswald hadn't killed Kennedy, Kennedy would be still alive".

account as a material implication. We cannot contradict this proposition either before or after the event, for it is true provided the man doesn't eat the cake, and before the event we have no reason to think he will eat it, and after the event we know he hasn't. Since he thinks nothing false, why do we dispute with him or condemn him?

Before the event we do differ from him in a quite clear way: it is not that he believes p, we $\bar{p}$; but he has a different degree of belief in q given p from ours; and we can obviously try to convert him to our view.[1] But after the event we both know that he did not eat the cake and that he was not ill; the difference between us is that he thinks that if he had eaten it he would have been ill, whereas we think he would not. But this is prima facie not a difference of degrees of belief in any proposition, for we both agree as to all the facts.

> [1] If two people are arguing 'If *p* will *q*?' and are both in doubt as to *p*, they are adding *p* hypothetically to their stock of knowledge and arguing on that basis about *q*; so that in a sense' If *p*, *q*' and 'If *p*, $\bar{q}$' are contradictories. We can say they are fixing their degrees of belief in *q* given *p*. If *p* turns out false, these degrees of belief are rendered void. If either party believes $\bar{p}$ for certain, the question ceases to mean anything to him except as a question about what follows from certain laws or hypotheses (Ramsey 1929: 246-47).

The procedure for evaluating conditional sentences described in this text is called *Ramsey's Test*. It inspired a suppositional analysis for conditionals, where 'If *p*, *q*' is interpreted as a hypothetical supposition that the antecedent *p* holds the believability of the consequent *q* under that supposition.

Like Ernest Adams and followers, some philosophers—focusing their attention on the concept of *degree of belief*—considered the remark above to apply probability logic[7] to conditional sentences. So, they interpreted Ramsey's Test via classical Bayesian conditionalization, inviting to measure the probability of "If *p*, *q*" by *conditioning* on *p*, identifying the probability of a conditional with the *conditional probability* on *q* given *p*.[8] This construal represents the reformulation of Ramsey's Test in a probabilistic thesis known as "*Equation*":

$P(p \longrightarrow q) = P(q \mid p)$ [where $P(p) > 0$][9]

[7] Ramsey's probability theory—called "logic of partial belief" by himself—is built on the idea that human beliefs cannot be based on an objective theory because they are connected to a whole set of epistemic attitudes through which people evaluate, choose and act. Ramsey did not mean to deny the existence of objective beliefs, but just to suggest to interpret human knowledge in terms of partial beliefs able to change in front of new evidences. The logic of partial belief wants to be just a way to calculate our beliefs such as subjective probabilities, establishing Bayes' theorem as the general rule to determine the probability update.

[8] Thomas Bayes was able to found an updating rule establishing how to adjust our *degree of belief* when we acquire new information. Indeed, the probability of any event *b* after learning that *a* is true (and nothing else) may be changed. How? The rule prescribed in Bayesian literature is to match the *posterior* probability of *b* ($P_{t1}(b)$) with the *prior* probability of *b* given *a* ($P_{t0}(b \mid a)$). This is just the *Bayesian conditionalization*—where $P_{t0}(b \mid a)$ is called *conditional probability*—and it can be formulated in this way: If $P_{t0}(a)>0$, then $P_{t1}(b) = P_{t0}(b \mid a)$.

[9] $P(p)>0$ because of *zero-intolerance* property of conditionals, according to whom if *p* has no chance of being true, there is not any conditional probability. In other words, nobody use a conditional sentence when know that the antecedent's probability is 0 (cf. Bennett 2003: 53-57).

Many philosophers and logicians advanced several proofs in support or against the Equation. A very important contribution is that of Adams, who had the worth of extending probabilistic logic to conditionals.

Since the mid-90s, Adams showed powerful arguments defending the probabilistic interpretation of Ramsey's Test, so that some philosophers started to talk about Ramsey-Adams Thesis.

Adams' analysis is restricted to *indicative* conditionals and started observing that propositional calculus's common use leads to fallacies when its application involves conditional sentences. So, the problem Adams raised concerns on how we have to use formal logic in conditional treatment. Indeed, he showed that many classically valid cases—in the sense that the premises cannot be true while its conclusion is false—are rejected (or at least doubtful) by common sense, leading to different kinds of fallacies.

Adams identified the trouble because when we deal with conditional statements, the term 'true' has no precise application. For this reason, he proposed to find a kind of validity that does not involve the notion of truth, with the intent to analyze conditional sentences from the point of view and not in terms of their truth conditions. So, he substituted the concept of classical validity with that of *reasonableness*, whose condition is:

> If an inference is reasonable, it should not be the case that on some occasion the assertion of its premises would be justified, but denial of its conclusion also justified (Adams 1975: 171).

So, while classical validity involves the notion of truth, the reasonableness concerns *justified assertability*, which is not a mathematical or scientific notion but rather a concept whose content is dictated by the assertion context. An assertion of a statement is justified if *what one knows* on that occasion gives us either the certainty or a high probability that the same statement will be true and win a bet on it. In the same way, denying that assertion is justified if we have either the certainty or a high probability that the statement will be false and the bet will be lost. Adams called the assertion *strictly justified* in case of certainty and *probabilistically justified* when the statement is just highly probable.

What about the assertion of "If *p*, *q*"? Adams converted the above notions in terms of conditional bets[10]—any bets on conditional statements—giving such a "betting" criterion of justification:

a. The assertion of a bettable conditional 'if *p* then *q*' is strictly justified on an occasion if what is known on that occasion makes it certain that either *p* is false or *q* is true; its denial either *p* is false, or *q* is false.
b. The assertion of a bettable conditional 'if *p* then *q*' is probabilistically justified on that occasion if what is known on that occasion makes it much more likely that *p* and *q* are both true than that *p* is true and *q* is false; its denial is probabilistically justified if it is much more likely that *p* is true and *q* is false than that *p* and *q* are both true.

[10] The notion of conditional bet was introduced first by de Finetti 1931. See also de Finetti 1937.

c. (Definition) The assertion and denial of a bettable conditional 'if p then q' are both vacuously probabilistically and strictly justified on an occasion if what is known on that occasion makes it certain that p is false (Adams 1975: 176-77).

Conditions for reasonableness of conditionals	$p \rightarrow q$	$\sim(p \rightarrow q) \equiv p \rightarrow \sim q$
Strictly justified	$\sim p \vee q \equiv P(\sim p \vee q)=1$	$\sim p \vee \sim q \equiv P(\sim p \vee \sim q)=1$
Probabilistically justified	$P(p \wedge q) > P(p \wedge \sim q)$	$P(p \wedge \sim q) > P(p \wedge q)$
Vacuously strictly and probabilistically justified	$\sim p \equiv P(p)=0$	$\sim p \equiv P(p)=0$

In the case of vacuous justification, one may assert the inference and its denial because the bet is not lost but just *called off*—according to the betting criterion. However, Adams pointed out that when we are sure the bet will be called off, we will not stake at all and we are asserting no indicative conditional. Indeed, in those cases in which we are sure about antecedent's falsity, we will use a subjunctive conditional—Adams' analysis does not address that.

Considering the notion of vacuous conditional, Adams reformulated the general condition for reasonableness of an inference saying that *it cannot be the case the assertion of its premises and the non-vacuous denial of its conclusion are both justified on the same occasion*.

Because of this notion of reasonableness, Adams showed that absurd cases classically valid—for example, the material conditional's fallacies—is not valid for the betting criterion of justification. Indeed, inferences like "If Brown wins the election, Smith will retire to private life. Therefore, if Smith dies before the election and Brown wins it, Smith will retire to private life" (Adams 1975: 166) are classically valid but not *reasonable*, because both the assertion of the premises and the negation of the conclusion are justified.

In Adams 1965, we can find an informal presentation of a reasonableness' criterion using the standard probability calculus. After solving several problems in conditional treatment, Adams concluded that this first analysis shows some critical limitation. For example, its application lacks with conditionals derived from suppositions and with compounds involving conditionals. So, he advanced the hypothesis that, maybe, assertable conditionals observe different logical laws, others from those of the standard propositional calculus.

Trying to overcome these limitations, in Adams 1966 (265-316) the original idea is formalized with some adjustment. First of all, the notion of "justified assertability" is now replaced by that of "high probability", and the criterion of reasonableness is consequently given simply substituting "true" with "high probability" in the definition of classical validity:

> an inference is *reasonable* just if its premises cannot have high probability while the conclusion has low probability (Adams 1966: 266).

Then, in Adams 1975, a consequence of this assumption is made explicit, introducing a technical term called "uncertainty" (u = 1– probability). So, in case of reasonable inference:

> […] *the uncertainty of its conclusion cannot exceed the uncertainty of its premises* (where uncertainty is here defined as the probability of falsity […]) (Adams 1975: 2).

The concept of uncertainty is fundamental in Adams because he could avoid any falsity concept in the definition of validity.

Therefore, in Adams's hypothesis, the strict connection between high probability and truth, characterizing unconditional statements, fails for conditionals.[11] Indeed, Adams advanced the idea according to which the probability of a conditional sentence should not be interpreted as the *probability of being true* but rather as *conditional probability*. Thus, Adams identified the Equation as a fundamental assumption of his analysis, making his thesis one of the most important arguments in defence of the Equation itself.

The idea that probability equals the probability of truth[12] entails that truth-conditional validity ensures reasonableness (*probabilistic-validity*) but, according to Adams, it holds *only* in case of factual propositions. Thus, if Adams' supposition is correct, the probability of a conditional cannot equal *in general* the probability it is true.

The fact that such a link between truth-conditional validity and p-validity holds for a factual proposition is easily demonstrable. For example, an inference like "It will either rain or snow tomorrow ($R \vee S$); it will not snow tomorrow ($\sim S$); therefore it will rain tomorrow (R)" (Adams 1975: 88) is classically valid when $R \vee S$ and $\sim S$ are true, and R is true too. Now, suppose that both $P(R \vee S)$ and $P(\sim S)$ equal 95% so that both $P(\sim R \,\&\!\sim S)$ and $P(S)$ are of 5%. Under these circumstances, the sum of the premises' uncertainties is 10%, thus $u(R) \leq 10\%$. This result means that, if the premises have *objective* probabilities of 95%, their conclusion has a probability of at least 90%, and this connection between objective probabilities and correct predictability makes that truth-conditional validity guarantees the probabilistic-validity.

The connection mentioned above cannot be shown in conditional sentences, and the truth-conditional validity lacks proof for reasonableness. For example, consider the conditional inference "If I eat those mushrooms, I will be poisoned" (Adams 1975: 89). The simple fact to not eat the mushrooms makes the inference materially true, but it is really difficult to say whether the assertion is right or wrong, so that the decision connected to it would be the best or the worst in terms of practical interest. Indeed, if the mushrooms are not poisoned, but delicious porcinis, and I decide not to eat them, my choice would not be right. This consideration confirms Adams' intuition, according to which the truth-conditional validity of a conditional inference does not prove its reasonableness—its probabilistic validity

[11] "The probability of a proposition is the same as the probability that it is true. […] What we want to argue next is that there is a much more radical divergences between the two soundness criteria in application to inferences involving conditional propositions, which is ultimately traceable to the failure of the probability equals probability of truth assumption in application to conditionals" (Adams 1975: 2).

[12] $P(\phi \Rightarrow \psi) = \frac{P(t_1)t_1(\phi\&\psi)+\cdots+P(t_n)t_n(\phi\&\psi)}{P(t_1)t_1(\phi)+\cdots+P(t_n)t_n(\phi)}$

cannot be guaranteed by classical validity—so that the rule according to which probability is the probability of truth fails with conditional inferences. Why? According to Adams, the explanation is that when we assert a conditional, we do not express a probability of truth, but nothing more than a conditional probability. This approach should explain many phenomena, like the mushrooms' example, in an easier way than standard probability. This remark is the reason for which Adams' Thesis is also known as Probability Conditional Thesis—(PCT): $P(p \rightarrow q) = P(q \mid p)$—and its relation with the Material Conditional Thesis—(MCT): $p \rightarrow q = p \supset q = {\sim}p \vee q$—is fixed by the *Conditional Deficit Formula* (CDF):[13]

$$P(p \supset q) - P(p \rightarrow q) = [1 - P(p \supset q)][\frac{P(\sim p)}{P(p)}].$$

Even though sometimes—when CDF is low—conditional probability can be inferred by material conditional, such a rule shows why generally they do not coincide at all.[14]

Although Adams' logic works pretty well with indicative conditionals, his thesis presents some limits because it does not always hold in common language. For example, inferences like "If it is sunny, then if it is my day off then I will go to the beach"—$p \rightarrow (q \rightarrow z)$—are excluded by Adams, but they may be asserted ordinarily, equalizing inferences like "If it is sunny and it is my day off, then I will go to the beach"—$(p \wedge q) \rightarrow z$—by the *Law of Importation*.[15] Also, inferences joining a standard proposition and a conditional one, like "Either I will stay at home or if Jane calls me then I will go to the cinema"—$p \vee (q \rightarrow z)$—are rejected by Adams, though they are really common in natural language.

However, our language is full of complications representing a real argument for rejecting a logic that seems to work well under many aspects. Perhaps, also, for this reason, Adams' hypothesis met several supporters. One of the most important is Dorothy Edgington (1986: 6-7), whose contributions helped make Adams' thesis one of the most shared in conditionals' field. Her arguments support either the Equation either Adams' conclusion that to accept $P(p \rightarrow q) = P(q \mid p)$ doubtless means to deny any truth conditions for conditional statements.[16]

[13] Adams 2005: 1-11.

[14] That these two kinds of probability do not coincide could be shown by a lot of example. One of these could be found in Adams 2005: 1-2.

[15] Law of Importation: $[p \rightarrow (q \rightarrow z)] \rightarrow [(p \wedge q) \rightarrow z]$. Vann McGee presented an argument supporting the ejection of iterated conditionals, reporting that when we say "If p, then if q then z" we are not accepting an iterated conditional, but rather a conditional with conjunctive antecedent, because what we have in mind is the conditional belief expressed by $(p \wedge q) \rightarrow z$. See McGee 1985: 462-71.

[16] However, in "On Conditionals" (2007: 180) Edgington accepts the view that when used to make conditional assertions, simple indicative conditionals may be considered true if both the antecedent and the consequent are true and false if the antecedent is true and the consequent false. "There is nothing comparably straightforward to say when the antecedent is false". According to Edgington, in any case, "[t]he 'true, false, neither' classification does not yield an interesting 3-valued logic or a promising treatment of compounds of conditionals".

3. Stalnaker Semantics for Conditional Statements

Moving from Adams' hypothesis appear to be a good start. When Robert Stalnaker developed his theory in 1968—using Kripke's models' technical machinery—tried to develop a truth-conditional semantics for conditionals—primarily for counterfactuals and covering the indicatives conditionals[17]—satisfying Ramsey-Adams Thesis. Indeed, in front of quite flawed theories—like the material implication analysis—Stalnaker thought to study Ramsey's Test, even though making some adjustments or trying to generalize it—given that Ramsey referred only to situations in which the agent has no idea about the antecedent's truth-value.

According to Stalnaker, this is the procedure for assessing credence to a conditional statement:

> First, add the antecedent (hypothetically) to your stock of beliefs; second, make whatever adjustments are required to maintain consistency (without modifying the hypothetical belief in the antecedent); finally, consider whether or not the consequent is then true (Stalnaker 1968: 41-55).

Once established belief-conditions for conditionals, how to fix truth-conditions? In this regard Stalnaker resorted to the Kripkean notion of a *possible world*, meant as the "ontological analogue of a stock of hypothetical beliefs" (Stalnaker 1968: 45) so that conditionals' truth-conditions for conditionals can be provided by an adaptation of truth-conditions settled by the possible world semantics:

> Consider a possible world in which A is true, and which otherwise differs minimally from the actual world. *"If A, then B" is true (false) just in case B is true (false) in that possible world"* (Stalnaker 1968: 45).

A theory of conditionals in terms of Kripke's models—developed independently of David Lewis—allows the transition from belief-conditions to truth conditions.

Then, Stalnaker built a probability system C2 by three steps, where each step represents a probability system itself, extension of the previous one.[18] By C2 Stalnaker developed parallelism between his semantics and the theory of conditional probability, showing that the theorems of C2 are nevertheless the valid sentences of Ramsey's Test.

Therefore, according to Adams' hypothesis about simple conditionals, Kripke's semantics works even though it yields some problems in the presence of compounds of conditionals. These problems are just the limit showed by the famous *Lewis' Triviality Result*, which shows that, if the probability of conditionals is the conditional probability $P(q \mid p)$ (as Adams guesses) and the probability of a sentence is always the probability of being true (as Stalnaker supports). There are sentences p and q such that the conditional "If p, q" whose probability of truth

[17] "The analysis was constructed primarily to account for counterfactual conditionals—conditionals whose antecedents are assumed by the speaker to be false—but the analysis was intended to fit conditional sentences generally, without regard to the attitudes taken by the speaker to antecedent or consequent or his purpose in uttering them, and without regard to grammatical mood in which the conditional is expressed" (Stalnaker 1976: 198).
[18] That argument could be found in Stalnaker 1970: 107-28.

coincides with $P(q \mid p)$ does not exist. In other words, the Triviality Result shows the incompatibility between the assumption that the probability of a proposition is *the probability it is true* and the *conditional probability* (Lewis 1981: 129-47), formalizing the divorce between Stalnaker's theory and the Equation.

4. Lewis' Triviality Result

In 1976 Lewis presented an argument, known as Triviality Result, showing the incompatibility between the assumption that the probability of a proposition is the probability it is true and the conditional probability. In such a way, the divorce between Stalnaker's theory and the Equation is formalized. There are many versions of the Triviality Result, but I prefer reporting the Lewis' original one.[19]

- Preliminaries:
  - Suppose we have a formal language containing at least the truth-functional connectives plus "→". Every connective could be used to compound any sentences in this language, whose truth-value is given in terms of possible worlds.
  - Define the conditional probability function in such a way:
    - $P(q \mid p) = P(q \wedge p) \mid P(p)$, if $P(p) > 0$.[20]
  - Assume the following standard probability laws:
    - $1 \geq P(p) \geq 0$.
    - If p and q are equivalent—both true at the same world—, then $P(p) = P(q)$.
    - If p and q are incompatible—both true at no world—, then $P(p \vee q) = P(p) + P(q)$.
    - If p is necessary—true in every world—, then $P(p)=1$.
  - Suppose to interpret "→" such that:
    - $P(p \rightarrow q) = P(q \mid p)$, if $P(p) > 0$, i.e. the probability of a conditional is its conditional probability.

    So that, if it holds, this holds too:
    - $P(p \rightarrow q \mid z) = P(q \mid p \wedge z)$, if $P(p \wedge z) > 0$.
- First Triviality Result:
  - Take $P(p \wedge q)$ and $P(p \wedge {\sim}q)$ both positive, so that $P(p)$, $P(q)$ and $P({\sim}q)$ are positive too. Now we have:
    - $P(p \rightarrow q) = P(q \mid p)$ holds by $P(p \rightarrow q) = P(q \mid p)$.
    - $P(p \rightarrow q \mid q) = P(q \mid p \wedge q) = 1$ and $P(p \rightarrow q \mid {\sim}q) = P(q \mid p \wedge {\sim}q) = 0$ hold by replacing z with q or ${\sim}q$ in $P(p \rightarrow q \mid z) = P(q \mid p \wedge z)$.

[19] A simpler version is that by Blackburn. See Blackburn 1986: 201-32.
[20] If $P(p) = 0$ then $P(q \mid p)$ remains undefined. A truthful speaker considers permissible to assert the indicative conditional $p \rightarrow q$ just in case $P(q \mid p)$ is sufficiently close to 1, i.e. only if $P(q \wedge p)$ is sufficiently greater that $P({\sim}q \wedge p)$. Lewis 1981: 129.

- For every sentence r, $P(r) = P(r \mid q) \cdot P(q) + P(r \mid {\sim}q) \cdot P({\sim}q)$ holds by expansion.
- Taking r as $p \rightarrow q$, we have:
  - $P(r) = P(q \mid p)$, by $P(p \rightarrow q) = P(q \mid p)$.
  - $P(r \mid q) = P(q \mid p \wedge q) = 1$ and $P(r \mid {\sim}q) = P(q \mid p \wedge {\sim}q) = 0$, by $P(p \rightarrow q \mid q) = P(q \mid p \wedge q) = 1$ and $P(p \rightarrow q \mid {\sim}q) = P(q \mid p \wedge {\sim}q) = 0$.

  So:

  - $P(q \mid p) = 1 \cdot P(q) + 0 \cdot P({\sim}q) = P(q)$ holds by substitution on $P(r) = P(r \mid q) \cdot P(q) + P(r \mid {\sim}q) \cdot P({\sim}q)$.
- *First conclusions:*
  - If $P(p \wedge q)$ and $P(p \wedge {\sim}q)$ are both positive then the propositions are probabilistically independent—that is absurd, though no contradictory.
  - Assigning standard true-values to any couple of propositions p and q, it derives that $P(q \mid p) = P(q)$, i.e. the conditional probability equals the probability of the consequent.

  Consequently:

  - Any language expressing a conditional probability is a *trivial language*.

- Second Triviality Result:
  - Suppose that "→" is a probability conditional for a class of probability functions closed under conditionalizing, and take any probability function P in the class and any sentences p and q such that $(p \wedge q)$ and $P(p \wedge {\sim}q)$ are both positive. Proceeding as before, we have again:
    - $P(q \mid p) = P(q)$.
  - Take three pairwise incompatible sentences q, z and r such that $P(q)$, $P(z)$ and $P(r)$ are all positive. Replacing the disjunction $(q \vee z)$ with p, we have that $P(p \wedge q)$ and $P(p \wedge {\sim}q)$ are both positive, but $P(q \mid p)$ does not equal $P(q)$. This result means there are no such three sentences.
  - *Second conclusions:*
    - P is a *trivial probability function* that never assigns positive probability to more than two incompatible alternative, so fixing at most four different values: $P(q)=1$ and $P(p)=1$—determining that $P(q \mid p) =1=P(q)$—, $P(q)=1$ and $P(p)=0$—so that $P(q \mid p)$ is an undefined number—, $P(q)=0$ and $P(p)=1$—determining that $P(q \mid p) = 0 =P(q)$—, $P(q)=0$ and $P(p)=0$—$P(q \mid p)$ is undefined again.

    Consequently:

    - For every class of probability function closed under conditionalizing, "→" cannot be a probability conditional unless the class consists entirely of trivial probability functions.

- Given that a probability function represents a possible belief system, and some of such systems are not trivial, then indicative conditionals cannot be considered probability conditionals for the whole class of probability functions.
- There is no guarantee that the probability of a conditional equals the corresponding conditional probability for all possible subjective probability functions, i.e. it is not a general rule that the absolute probability of a conditional proposition equals the probability of its consequence on condition of its antecedent.

It is quite clear that the second Triviality Result logically entails the first one, the reason for which Lewis' argument is generally called just "Triviality Result".

Even if Stalnaker firstly agreed with the idea the probability of a conditional equals its conditional probability, in front of Lewis' Result he seems to give up the Equation and, in general, a suppositional view. Alternatively, Adams held his thesis inviting to consider conditionals, not as standard propositions, but as particular linguistic constructions always lacking truth values and conditions. However, I want to point out that Adams denied any truth-values and conditions for indicative conditionals only in 1975, after knowing the problems raised by Lewis' Triviality Result. Of course, Adams proposed to analyze conditional inferences in terms of probability since his first approach, but I think this is different from the total denial of truth conditions. I am not completely sure he would have advanced such a "drastic" solution if any Triviality Result would not have been possible.

Lewis himself, although not explicitly, disagreed with Adams' conclusion. He thought that "fortunately a more conservative hypothesis is at hand" (Lewis (1981: 137): Grice's theory. One may identify its conversational rules with those special rules useful to understand why assertability goes with conditional probability. So, Lewis suggested that we should start from something already known, rather than run into those complications Adams' hypothesis requires. For this reason, he adopted the material conditional's truth conditions, explaining the discrepancy between its probability of truth and its assertability by a Gricean implication. In a first moment, Lewis talked about a *conversational* implication, but then he opted for Jackson's theory, in favour of a *conventional* one (Lewis 1987: 151-56):

> An indicative conditional is a truth-functional conditional that conventionally implicates robustness with respect to the antecedent. Therefore, an indicative conditional with antecedent A and consequent C is assertable iff (or to the extent that) the probabilities $P(A \supset C)$ and $P(A \supset C/A)$ both are high. If the second is high, the first will be too; and the second is high iff $P(C/A)$ is high; and that is the reason why the assertability of indicative conditionals goes by the corresponding conditional probability (Lewis 1987: 153).

In any case, the real problem of Adams' conclusion concerns compound of sentences. Indeed, even if he was right, and conditionals with truth-valued antecedent and consequent would be governed only by assertability rules—different from standard probability rules—what about those conditionals compounded of conditional antecedent and consequent, lacking themselves of any value, condition and probability of truth? Adams should admit that the common idea is that we can know the truth-conditions of molecular sentences once we know their sub-sentences' truth-conditions. However, how could it be possible when sub-sentences lack truth-conditions? In that case, we need something different from those assertability rules, because, in front of this new evidence, they cannot show how compound sentences work. We need at least a new semantics containing special rules or anything else able to explain them.

5. An Alternative Way according to de Finetti

Though not without difficulty, Adams' approach represents one of the most successful theories of conditionals. However, the question I raise is this:

Does the Triviality Result lead to Adams' conclusion to deny any truth conditions and values for indicative conditionals?

In other words:

Might it exist an alternative way to avoid Lewis' Result following the Equation?

To answer such a question, I will introduce de Finetti's logic, a kind of three-valued logic, called "Logic of Tri-events", which seems to avoid the Triviality Result—even though it is no free from trouble.

Bruno de Finetti is known to be the founder—together with Ramsey, but independently—of the subjective interpretation of probability. He developed his analysis in terms of a betting system: probability is a special case of prevision corresponding to a bet's price. In case of a conditional bet, that is a gamble on a proposition q supposed that an event p happens, its price will equal the conditional probability of $q \mid p$, i.e. a conditional bet coincides with a suppositional conditional.[21]

According to de Finetti, a conditional bet on q supposed that p will be (i) win when either p either q are true, (ii) lost when p is true and q false, (iii) called off when p is false. Therefore, he suggested to assign to $q \mid p$ a truth-value just in case of win or loss, and to consider it null—neither true nor false—when the bet is called for. In such a way a conditional event appears as a three-valued proposition, called "tri-event".

In 1935, de Finetti proposed a kind of logic of conditional events, known as "Logic of Tri-events", consisting of a three-valued logic expressing the question concerning conditional probabilities.[22] The basic idea is that the act to assume a standard two-valued logic is just a conventional issue: propositions are not true or false because of *a priori* principle, but because we conventionally decided to call "propositions" those logical entities needing of a "yes" or "no" as the answer. However, if we agreed on assume three values, we could have an analogue of standard logic, but with more values, differing just in a purely formal way.

In the Logic of Tri-events, the third value is not, strictly speaking, a value like "true" or "false". We have to consider a third possible attitude that someone

[21] De Finetti made use of the notion of "conditional expectation"—$P(X \mid H) = P(X \wedge H)/P(H)$—that allows to interpret the conditional probability such as the expected conditional value of the prize of a conditional bet. This is important because Stalnaker & Jeffrey and McGee made the mistake to consider the value of a conditional bet such as the absolute expectation value of its prize, interpreting a called off bet such as zero profit. But, put in this way, in the Bayesian theory a called off bet is something which remains unchanged to positive linear transformations—there is not any zero equipped of an intrinsic value. In de Finetti's view the gain of a called off bet is indefinite.

[22] Bruno de Finetti 1935: 181-90.

can adopt toward a proposition when he doubts answering "yes" or "no". In other words, this third value is void—or *null*—and one can understand it as a gap. However, a null event is something different from an indeterminate event á la Lukasiewicz—whose truth conditions are *unknown*. Rather, de Finetti meant an event whose truth-values true or false are not satisfied. We can find several de Finetti's papers talking about this third value, and he has never changed his interpretation about that. It is particularly interesting the passage in which he identified a null event with an "aborted event":

> If a distinction results in being incomplete, no harm was done: it would mean that besides "true" and "false" events I would also have "null" events, or, so to speak, aborted events. As a matter of fact, it is sometimes useful to consider explicitly and intentionally from the very start such a "tri-event" (especially, as will be seen later, for probability theory). If, for instance, I say: "supposing that I miss the train, I shall live by car", I am formulating a "tri-event", which will be either true or false if, after missing the train, I leave by car or not, and it will be null if I do not miss the train.[23]

One may expand standard logic's truth-tables to include the null value in such a way:

p	q	$\sim p$	$p \vee q$	$p \wedge q$	$p \supset q$[24]	$q \mid p$
T	**T**	F	T	T	T	T
T	**F**	F	T	F	F	F
T	**N**	F	T	N	N	N
F	**T**	T	T	F	T	N
F	**F**	T	F	F	T	N
F	**N**	T	N	F	T	N
N	**T**	N	T	N	T	N
N	**F**	N	N	F	N	N
N	**N**	N	N	N	N	N

While conjunction and disjunction coincide with those proposed by Lukasiewicz' three-valued logic, conditioning is the new truth-function introduced by de Finetti. So, the real innovation consists just in the truth-functional conditioning connective "|".

According to de Finetti, such a kind of logic should help us to manage those troubles due to a two-valued analysis, with the advantage that one may translate every proposition in terms of standard logic—given that every tri-event is a simply formal representation of pairs of ordinary events.[25] Indeed, a return from the Logic of Tri-events to the standard two-valued logic is possible by introducing

[23] Translation by Alberto Mura of Bruno de Finetti 1934/2006: 103, in Mura 2009: 204.
[24] This material conditional is today known as "Kleene's strong material implication", because independently proposed later by Kleene in 1938. See Kleene 1938: 150-55.
[25] However, it should be pointed out that the algebra of such a pairs of ordinary events—isomorphic to the trievents' algebra—is not Boolean. It is rather a distributive lattice that does not admit a unique complement—it means it does not hold CEM.

two operations: *thesis* (T) and *hypothesis* (H).[26] $T(X)$ means "X is true" and $H(X)$ means "X is not null":[27]

X	$T(X)$	$H(X)$
T	T	T
F	F	T
N	F	F

The above truth-table shows it holds that $X = T(X) \mid H(X)$, i.e. every tri-event ϕ is true, given that it is not null. This result is known as "de Finetti's Decomposition Theorem".[28]

Given that every tri-event can be represented by any conditional event $q \mid p$—where p and q are ordinary events—, for the Decomposition Theorem it holds that $q \mid p = T(q \mid p) \mid H(q \mid p)$. Looking at the truth-table of "$\mid$", excluding those cases where p and q are aborted events, the Decomposition Theorem leads to two important consequences:

- $q \mid p$ is true if and only if both p and q are true—$T(q \mid p) = p \wedge q$.
- $q \mid p$ is not null if and only if p is a tautology—$H(q \mid p) = p$.

$\boldsymbol{p}$	$\boldsymbol{q}$	$\boldsymbol{q \mid p}$
T	**T**	**T**
T	**F**	**F**
F	T	N
F	F	N

Thus, it results that $q \mid p = T(q \mid p) \mid H(q \mid p) = (p \wedge q) \mid p$.

If p is not a tautology, it means it could be false, so that $q \mid p$ is not an ordinary event. Consequently, an ordinary event is nothing less than a particular case of a tri-event when p is a tautology. Therefore, "to introduce the notion of conditional probability is to extend the definition of $P(X)$ from the field of ordinary events, X, to the field of tri-events" (de Finetti 1935: 184-85). In Mura 2009 (214-16) we can find two methods to obtain such extension.

In conclusion, de Finetti's analysis shows that every probability function defined on a Boolean algebra of ordinary events can be univocally extended to the whole tri-events lattice, so that, given two standard proposition p and q, the probability of the tri-event $q \mid p$ equals the ratio between the probability of $p \wedge q$ and the probability of p. Consequently, "$\mid$" appears such as a connective satisfying the Equation but with the advantage of avoiding Lewis' Triviality Result—because $q \mid p$ is not an ordinary event, but a three-valued proposition.

[26] The rule of Thesis and Hypothesis is just that of allowing a conversion into standard logic. So, technically, they are not operators belonging to the logic of Trievents. About this, see Mura 2009: 207-209.
[27] In terms of betting system, "the 'thesis' of the tri-event, is the case in which one has established that the bet is won; the 'hypothesis' the case in which one has established that the best is in effect" (Bruno de Finetti 1935: 186).
[28] So called by Alberto Mura. See Mura 2009: 208.

Although de Finetti's account can represent a way to avoid trivialization conserving the Equation, it is not free from problems, making it unable to provide a right semantic for conditional statements. For example, despite increasing relation to some aspects, the correspondence between logic and probability loses some properties on the other side. Among them, in de Finetti's account, $\phi \mid \phi$ is not a tautology, but a *quasi-tautology*, because although it is not false, it can be either true or null. So, given any p and q and any probability function P, if $P(p) = P(q)$ but $p \mid p$ is not truth-functionally equivalent to $q \mid q$, then $p \neq q$. In other words, it does not work the propriety according to which, when two tri-events have, for every probability function, the same probability, the respective propositions are logically equivalent. This result means there are various tri-events, to which every probability function assigns probability 1, but not logically equivalent. Similarly, any de Finetti's contradiction is a *quasi-contradiction*,[29] given that it cannot be true, but can be either false or null. So, it is easy to catch that some elements can be quasi-tautologies and quasi-contradictions simultaneously.

Some of de Finetti's approach seems to be overcome by a modified tri-events approach, developed by Alberto Mura.

Mura's proposal was presented first as "Semantics of Hypervaluations" and then improved as "Theory of Hypervaluated Trievents". Mura gave a modified account of de Finetti's tri-events—escaping different arguments against the original tri-events—with the intent of providing a new semantic for Adams' conditional logic. I will claim that Mura's account can be a good candidate for a semantic of indicative conditionals, in perfect harmony with Adams analysis. Indeed, the Theory of Hypervaluated Tri-events incorporates Adams' p-entailment, allowing an extension of it for all tri-events—including compounds of conditionals. In this way, we are no more obligated to reject any truth conditions for conditionals.

Moreover, Mura proposed a generalization of the Theory of Hypervaluated Trievents to catch counterfactual conditionals by introducing a new variable K representing the corpus of total beliefs.

In conclusion, I wanted to evidence that conditional issue is not a closed topic and that different additional ways can be investigated. For example, any theory developed on a three-value logic might be a good solution that deserves to be inquired, letting us still aspire to a unified theory for conditional sentences.

References

Adams, E.W. 1966, "Probability and the Logic of Conditionals", in Hintikka, J. and Suppes, P. (eds.), *Aspects of Inductive Logic*, Amsterdam: North Holland, 265-316.

Adams, E.W. 1970, "Subjunctive and Indicative Conditionals", *Foundations of Language*, 6, 89-94.

Adams, E.W. 1975, *The Logic of Conditionals. An Application of Probability to Deductive Logic*, Dordrecht: Reidel.

[29] Both terms of "quasi-tautology" and "quasi-contradiction" due to Bergman. See Bergman 2008: 85-86.

Adams, E.W. 2005, "What Is at Stake in the Controversy over Conditional", in Kern-Isberner, G., Rödder, W. and Kulmann F. (eds.), *Conditionals, Information, and Inference, WCII 2002*, Berlin-Heidelberg: Springer, 1-11.

Bennett, J. 2003, *A Philosophical Guide to Conditionals*, Oxford: Clarendon Press, 53-57.

Bergman, M. 2008, *An Introduction to Many-Valued and Fuzzy Logic*, Cambridge: Cambridge University Press.

Blackburn, S. 1986, "How Can We Tell Whether a Commitment Has a Truth Condition?", in Travis, C. (ed.), *Meaning and Interpretation*, Oxford: Blackwell, 201-32.

de Finetti, B. 1931, "Sul significato soggettivo della probabilità", *Fundamenta Mathematicae*, 17, 298-329.

de Finetti, B. 1934/2006, *L'invenzione della verità*, Milano: Cortina.

de Finetti, B. 1935, "The Logic of Probability", *Philosophical Studies*, 77, 181-90.

de Finetti, B. 1937, "La Prévision: Ses Lois Logiques, Ses Sources Subjectives", *Annales de l'Institut Henri Poincaré*, 7, 1-68. Translated as "Foresight: Its Logical Laws, Its Subjective Sources", in Kyburg, H.E. and Smokler, H.E. 1980 (eds.), *Studies in Subjective Probability*, New York: Robert E. Krieger.

Edgington, D. 1986, "Do Conditionals Have Truth Conditions?", *Crítica*, 18, 6-7.

Edgington, D. 2007, "On Conditionals", in Gabbay, D.M. and Guenthner, F. (eds.), *Handbook of Philosophical Logic*, 2nd Edition, Vol. 14, Berlin: Springer, 127-221.

Kleene, S.C. 1938, "On Notation for Ordinal Numbers", *The Journal of Symbolic Logic*, 3, 150-55.

Lewis, D. 1973, *Counterfactuals*, Oxford: Blackwell.

Lewis, D. 1981. "Probabilities of Conditionals and Conditional Probabilities", in Harper, W.L., Stalnaker, R. and Pearce, G. (eds.), *Ifs: Conditionals, Belief, Decision, Chance, and Time*, Dordrecht & Boston: Reidel, 129-47.

Lewis, D. 1987, *Philosophical Paper*, Vol. II, New York: Oxford University Press.

Mackie, J.L. 1973, *Truth, Probability, and Paradox*, Oxford: Clarendon Press.

McGee, V. 1985, "A Counterexample to Modus Ponens", *The Journal of Philosophy*, 82, 462-71.

Mura, A. 2009, "Probability and the Logic of de Finetti's Trievents", in Galavotti M.C. (ed.), *Bruno de Finetti, Radical Probabilist*, London: College Publications, 201-42.

Ramsey, F.P. 1929, "General Propositions and Causality", in Braithwaite, R.B. (ed.), *Foundations of Mathematics and Other Logical Essays*, London: Routledge & Kegan 1931, 246-47.

Stalnaker, R.C. 1968, "A Theory of Conditionals", in Harper, W.L., Stalnaker, R. and Pearce, G. (eds.), *Ifs: Conditionals, Belief, Decision, Chance, and Time*, Dordrecht & Boston: Reidel 1981, 41-55.

Stalnaker, R.C. 1970, "Probability and Conditionals", in Harper, W.L., Stalnaker, R. and Pearce, G. (eds.), *Ifs: Conditionals, Belief, Decision, Chance, and Time*, Dordrecht & Boston: Reidel 1981, 107-28.

Stalnaker, R.C. 1976, "Indicative Conditionals", in Harper, W.L., Stalnaker, R. and Pearce, G. (eds.), *Ifs: Conditionals, Belief, Decision, Chance, and Time*, Dordrecht & Boston: Reidel 1981, 193-210.

Woods, M., Wiggins, D. and Edgington, D. 1997, *Conditionals*, Oxford: Clarendon Press.

On Compound and Iterated Conditionals

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Abstract

We illustrate the notions of compound and iterated conditionals introduced, in recent papers, as suitable conditional random quantities, in the framework of coherence. We motivate our definitions by examining some concrete examples. Our logical operations among conditional events satisfy the basic probabilistic properties valid for unconditional events. We show that some, intuitively acceptable, compound sentences on conditionals can be analyzed in a rigorous way in terms of suitable iterated conditionals. We discuss the Import-Export principle, which is not valid in our approach, by also examining the inference from a material conditional to the associated conditional event. Then, we illustrate the characterization, in terms of iterated conditionals, of some well known p-valid and non p-valid inference rules.

Keywords: Coherence, Conditional events, Conditional random quantities, Conjunction, Disjunction, Iterated conditional, Inference rules, p-validity, p-entailment, Import-Export principle.

1. Introduction

Let us imagine an experiment where you flip a coin twice; then, let us consider the conjunction

"*the outcome of the 1st flip is head and the outcome of the 2nd flip is head*".

By defining the events A = "*the outcome of the 1st flip is head*" and B = "*the outcome of the 2nd flip is head*", we denote by $A \wedge B$, or simply by AB, the previous conjunction, which is true when both A and B are true, and false when A or B is false. If you judge $P(AB) = p$, then in a bet on AB you agree to pay, for instance, p by receiving 1, or 0, according to whether AB turns out to be *true*, or *false*, respectively.

What is the "logical value" of AB when the outcome of the first coin is head and the second coin stands up? We cannot answer because the event B is neither true nor false.

Moreover, what happens of the bet? Cases like this are not considered when assessing $P(AB)$ (they are assumed to be impossible, or to have zero probability).

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Usually, the bet is called off and you receive back the paid amount p. Actually, by introducing the events H =*the outcome of the 1st flip is head or tail* and K =*the outcome of the 2nd flip is head or tail*, we realize that when evaluating $P(AB)$, in fact we are evaluating $P(AB|HK)$, under the implicit assumption that $P(HK) = 1$. Indeed, by observing that $P(AB|\overline{HK}) = 0$, it follows that

$$P(AB) = P(AB|HK)P(HK) + P(AB|\overline{HK})P(\overline{HK}) = P(AB|HK)P(HK),$$

and when $P(HK) = 1$ it holds that $P(AB) = P(AB|HK)$. On the contrary, when $P(HK) < 1$, one has $P(AB) < P(AB|HK)$, in which case the paid amount $P(AB)$ is less than the amount (that should be paid) $P(AB|HK)$. Moreover,as $\Omega = HK \vee H\overline{K} \vee \overline{H}K \vee \overline{H}\overline{K} = HK \vee \overline{HK}$, the event $\overline{HK}$ is the disjunction of three logical cases, that is $\overline{HK} = H\overline{K} \vee \overline{H}K \vee \overline{H}\overline{K}$, and such cases could be judged to be not similar. In particular, $\overline{H}\overline{K}$ appears different from the other two cases. What should be a general approach to this kind of more complex situations? We observe that, in order to manage these cases, the two sentences

the outcome of the 1st flip is head,

the outcome of the 2nd flip is head

should be written, respectively, as the conditional sentences

the outcome of the 1st flip is head, given that it is head or tail,
the outcome of the 2nd flip is head, given that it is head or tail;

that is, the events A, B should be replaced by the conditional events $A|H$, $B|K$. Moreover, the conjunction AB should be written as a suitable conjoined conditional $(A|H) \wedge (B|K)$. Based on the theories of de Finetti (1936, 1980) and Ramsey (1990), we look at the conditional *if H then A* as the conditional event $A|H$, hence in our approach it is satisfied *the Equation* (Edgington 1995), or *Conditional Probability Hypothesis* (see, e.g., Sanfilippo, Pfeifer, Over et al. 2018; Sanfilippo, Gilio, Over et al. 2020; Cruz 2020; Over and Cruz 2021), which establishes that the probability of a conditional coincides with the probability of the associated conditional event: $P(\textit{if } H \textit{ then } A) = P(A|H)$. Then, the conditional events $A|H$ and $B|K$ are associated with the following two conditionals:

1) *if the outcome of the 1st flip is head or tail, then it is head,*
2) *if the outcome of the 2nd flip is head or tail, then it is head.*

Moreover, by defining *valid* the flip when "the coin does not stand, or similar things", that is when "the outcome of the flip is head or tail", the conjunction $(A|H) \wedge (B|K)$ can be interpreted as the **conjoined conditional**

(if the outcome of the 1st flip is head or tail, then it is head) and *(if the outcome of the 2nd flip is head or tail, then it is head*.

What are the possible values of this conjoined conditional $(A|H) \wedge (B|K)$? The same analysis can be done for the disjunction $(A|H) \vee (B|K)$.

The problem of suitably defining logical operations among conditional events has been largely discussed by many authors (see, e.g., Baratgin, Politzer, Over et al. 2018; Benferhat, Dubois and Prade 1997; Coletti, Scozzafava and Vantaggi

2013, 2015; Douven, Elqayam, Singmann et al. 2019; Flaminio, Godo and Hosni 2020; Goodman, Nguyen and Walker 1991; Kaufmann 2009; Mura 2011; McGee 1989; Nguyen and Walker 1994). In a pioneering paper, written in 1935, de Finetti (1936) proposed a three-valued logic for conditional events, also studied by Lukasiewicz. Moreover, several authors have given many contributions to research on three-valued logics and compounds of conditionals (see, e.g.,Edgington 1995; McGee 1989; Milne 1997). Coherence-based probability logic has been recently discussed in Pfeifer 2021.

Usually, conjunction and disjunction of conditionals have been defined as suitable conditionals (see, e.g., Adams 1975; Calabrese 1987, 2017; Ciucci and Dubois 2012, 2013; Goodman, Nguyen and Walker 1991). However, in this way many classical probabilistic properties are lost. In particular, differently from the case of unconditional events, the lower and upper probability bounds for the conjunction of two conditional events are no more the Fréchet-Hoeffding bounds. This aspect has been studied in Sanfilippo 2018.

A more general approach where the result of conjunction or disjunction is no longer a three-valued object has been given in Kaufmann 2009; McGee 1989. In recent years, a related theory (Gilio and Sanfilippo 2013a, 2013b, 2014, 2017, 2019, 2020), with some applications (Gilio, Over, Pfeifer et al. 2017; Gilio, Pfeifer and Sanfilippo 2020; Sanfilippo, Gilio, Over et al. 2020; Sanfilippo, Pfeifer and Gilio 2017; Sanfilippo, Pfeifer, Over et al. 2018), has been developed in the setting of coherence, where conditioning events with zero probability are properly managed. In these papers the notions of compound and iterated conditionals are defined as suitable *conditional random quantities* with a finite number of possible values in the interval $[0,1]$. Within our approach the basic properties, valid for unconditional events, are preserved. In particular:

- the inequality $AB \leq \min\{A,B\}$ becomes $(A|H) \wedge (B|K) \leq \min\{A|H, B|K\}$ (the inequality $A \vee B \geq \max\{A,B\}$ becomes $(A|H) \vee (B|K) \geq \max\{A|H, B|K\}$);
- the Fréchet-Hoeffding lower and upper prevision bounds for the conjunction of conditional events still hold (Gilio and Sanfilippo 2021a);
- De Morgan's Laws are satisfied (Gilio and Sanfilippo 2019);
- the inclusion-exclusion formula for the disjunction of conditional events is valid (Gilio and Sanfilippo 2020); for instance, the formula $P(E_1 \vee E_2) = P(E_1) + P(E_2) - P(E_1E_2)$ becomes $\mathbb{P}[(E_1|H_1) \vee (E_2|H_2)] = P(E_1|H_1) + P(E_2|H_2) - \mathbb{P}[(E_1|H_1) \wedge (E_2|H_2)]$ (Gilio and Sanfilippo 2014);
- we can introduce the set of (conditional) constituents, with properties analogous to the case of unconditional events (Gilio and Sanfilippo 2020);
- by exploiting conjunction we obtain a characterization of the probabilistic entailment of Adams (Adams 1975) for conditionals (Gilio and Sanfilippo 2019); moreover, by exploiting iterated conditionals, the p-entailment of $E_3|H_3$ from a p-consistent family $\{E_1|H_1, E_2|H_2\}$ is characterized by the property that the iterated conditional $(E_3|H_3)|((E_1|H_1)\wedge(E_2|H_2))$ is constant and coincides with 1 (Gilio, Pfeifer and Sanfilippo 2020).

In our theory of compound and iterated conditionals, as in Adams 1975, Kaufmann 2009 and differently from McGee 1989, the Import-Export principle is not valid. As a consequence, as shown in Gilio and Sanfilippo 2014 (see also Sanfilippo, Gilio, Over et al. 2020; Sanfilippo, Pfeifer, Over et al. 2018), we avoid Lewis' triviality results (Lewis 1976). Probabilistic modus ponens has been generalized to conditional events (Sanfilippo, Pfeifer and Gilio 2017); moreover, one-premise and two-premise centering inferences has been studied in Gilio, Over, Pfeifer et al. 2017; Sanfilippo, Pfeifer, Over et al. 2018. In Sanfilippo, Gilio, Over et al. 2020 some intuitive probabilistic assessments discussed in Douven and Dietz 2011 have been explained, by making some implicit background information explicit.

The paper is organized as follows: In Section 2. we recall some basic notions and results on coherence, conditional events, and conditional random quantities. Moreover, we recall the definitions of p-consistency and p-entailment in the setting of coherence. Then we illustrate the notions of conjoined, disjoined and iterated conditional. In Section 3. we recall some well known probabilistic properties valid for unconditional events. Then, we show that these properties continue to hold when replacing events by conditional events. In Section 4. we show that some compound sentences on conditionals, which seem intuitively acceptable, can be analyzed in a rigorous way in terms of iterated conditionals. Moreover, we discuss the Import-Export principle, by also examining the iterated conditional $(A|H)|(\overline{H} \vee A)$. Then we illustrate, in terms of suitable iterated conditionals, several well known, p-valid and non p-valid, inference rules. In Section 5. we give some conclusions.

2. Preliminary Notions and Results

In our approach events represent uncertain facts described by (non ambiguous) logical propositions. An event A is a two-valued logical entity which is either *true*, or *false*. The indicator of an event A is a two-valued numerical quantity which is 1, or 0, according to whether A is true, or false, respectively. We use the same symbol to refer to an event and its indicator. We denote by Ω the sure event and by $\emptyset$ the impossible one (notice that, when necessary, the symbol $\emptyset$ will denote the empty set). Given two events A and B, we denote by $A \wedge B$, or simply by AB, the intersection, or conjunction, of A and B, as defined in propositional logic; likewise, we denote by $A \vee B$ the union, or disjunction, of A and B. We denote by $\overline{A}$ the negation of A. Of course, the truth values for conjunctions, disjunctions and negations are defined as usual. Given any events A and B, we simply write $A \subseteq B$ to denote that A logically implies B, that is $A\overline{B} = \emptyset$, which means that A and $\overline{B}$ cannot both be true.

2.1 Conditional Events, Coherence, and Conditional Random Quantities

Given two events E, H, with $H \neq \emptyset$, the conditional event $E|H$ is defined as a three-valued logical entity which is *true*, or *false*, or *void*, according to whether EH is true, or $\overline{E}H$ is true, or $\overline{H}$ is true, respectively. In the betting framework, to assess $P(E|H) = x$ amounts to say that, for every real number s, you are willing to pay an amount sx and to receive s, or 0, or sx, according to whether EH is true, or $\overline{E}H$ is true, or $\overline{H}$ is true (bet called off), respectively. Then for the random gain $G = sH(E - x)$, the possible values are $s(1 - x)$, or $-sx$, or 0, according to whether EH is true, or $\overline{E}H$ is true, or $\overline{H}$ is true, respectively. More generally speaking, consider a real-valued function $P : \mathcal{K} \to \mathbb{R}$, where $\mathcal{K}$ is an arbitrary (possibly not finite) family of conditional events. Let $\mathcal{F} = \{E_1|H_1, \dots, E_n|H_n\}$ be a family of conditional events, where $E_i|H_i \in \mathcal{K}$, $i = 1, \dots, n$, and let $\mathcal{P} = (p_1, \dots, p_n)$ be the vector of values $p_i = P(E_i|H_i)$, where $i = 1, \dots, n$. We denote by $\mathcal{H}_n$ the disjunction $H_1 \vee \dots \vee H_n$. With the pair $(\mathcal{F}, \mathcal{P})$ we associate the random gain $G = \sum_{i=1}^{n} s_i H_i (E_i - p_i)$, where $s_1, \dots, s_n$ are n arbitrary real numbers. G represents the net gain of n transactions. Let $\mathcal{G}_{\mathcal{H}_n}$ denote the set of possible values of G restricted to $\mathcal{H}_n$, that is, the values of G when at least one conditioning event is true.

Definition 1

The function P defined on $\mathcal{K}$ is coherent *if and only if, for every integer n, for every finite subfamily $\mathcal{F} = \{E_1|H_1, \dots, E_n|H_n\}$ of $\mathcal{K}$ and for every real numbers $s_1, \dots, s_n$, it holds that:* $\min \mathcal{G}_{\mathcal{H}_n} \leq 0 \leq \max \mathcal{G}_{\mathcal{H}_n}$.

Intuitively, Definition 1 means in betting terms that a probability assessment is coherent if and only if, in any finite combination of n bets, it cannot happen that the values in $\mathcal{G}_{\mathcal{H}_n}$ are all positive, or all negative (*no Dutch Book*).

We denote by X a *random quantity*, that is an uncertain real quantity, which has a well determined but unknown value. We assume that X has a finite set of possible values. Given any event $H \neq \emptyset$, agreeing to the betting metaphor, if you assess that the prevision of *"X conditional on H"* (or short: *"X given H"*), $\mathbb{P}(X|H)$, is equal to μ, this means that for any given real number s you are willing to pay an amount μs and to receive sX, or μs, according to whether H is true, or false (bet called off), respectively. In particular, when X is (the indicator of) an event A, then $\mathbb{P}(X|H) = P(A|H)$. Definition 1 can be generalized to the case of prevision assessments on a family of conditional random quantities (see, e.g., Gilio and Sanfilippo 2020). Given a conditional event $A|H$ with $P(A|H) = x$, the indicator of $A|H$, denoted by the same symbol, is

$$A|H = AH + x\overline{H} = AH + x(1 - H) = \begin{cases} 1, & \text{if } AH \text{ is true,} \\ 0, & \text{if } \overline{A}H \text{ is true,} \\ x, & \text{if } \overline{H} \text{ is true.} \end{cases} \quad (1)$$

The third value of the random quantity $A|H$ (subjectively) depends on the assessed probability $P(A|H) = x$. When $H \subseteq A$ (i.e., $AH = H$), it holds that $P(A|H) = 1$; then, for the indicator $A|H$ it holds that

$$A|H = AH + x\overline{H} = H + \overline{H} = 1, \text{ (when } H \subseteq A). \quad (2)$$

Likewise, if $AH = \emptyset$, it holds that $P(A|H) = 0$; then

$$A|H = 0 + 0\overline{H} = 0, \text{ (when } AH = \emptyset).$$

Given a random quantity X and an event $H \neq \emptyset$, with $P(X|H) = \mu$, in our approach, likewise formula (1), the conditional random quantity $X|H$ is defined as

$$X|H = XH + \mu\overline{H}.$$

(For a discussion on this extended notion of a conditional random quantity see, e.g., Gilio and Sanfilippo 2014; Sanfilippo, Gilio, Over et al. 2020.)

Remark 1

Given a conditional random quantity $X|H$ and a prevision assessment $\mathbb{P}(X|H) = \mu$, if conditionally on H being true X is constant, say $X = c$, then by coherence $\mu = c$.

The result below establishes some conditions under which two conditional random quantities $X|H$ and $Y|K$ coincide (Gilio and Sanfilippo 2014, Theorem 4).

Theorem 1

Given any events $H \neq \emptyset$ and $K \neq \emptyset$, and any random quantities X and Y, let Π be the set of the coherent prevision assessments $\mathbb{P}(X|H) = \mu$ and $\mathbb{P}(Y|K) = \nu$.

(i) Assume that, for every $(\mu, \nu) \in \Pi$, the values of $X|H$ and $Y|K$ always coincide when $H \vee K$ is true; then $\mu = \nu$ for every $(\mu, \nu) \in \Pi$.

(ii) For every $(\mu, \nu) \in \Pi$, the values of $X|H$ and $Y|K$ always coincide when $H \vee K$ is true if and only if $X|H = Y|K$.

2.2 Probabilistic Consistency and Entailment

We recall below the notion of logical implication of Goodman and Nguyen 1988 for conditional events (see also Gilio and Sanfilippo 2013d).

Definition 2

Given two conditional events $A|H$ and $B|K$ we define that $A|H$ logically implies *$B|K$ (denoted by $A|H \subseteq B|K$) if and only if AH logically implies BK and $\bar{B}K$ logically implies $\bar{A}H$; i.e., $AH \subseteq BK$ and $\bar{B}K \subseteq \bar{A}H$.*

A generalization of the Goodman and Nguyen logical implication to conditional random quantities has been given in Pelessoni and Vicig 2014.

The notions of p-consistency and p-entailment of Adams 1975 were formulated for conditional events in the setting of coherence in Gilio and Sanfilippo 2013d (see also Biazzo, Gilio, Lukasiewicz et al. 2005; Gilio 2002; Gilio and Sanfilippo 2013c).

Definition 3

Let $\mathcal{F}_n = \{E_i|H_i\,,\ i = 1,\ldots,n\}$ be a family of n conditional events. Then, $\mathcal{F}_n$ is p-consistent *if and only if the probability assessment $(p_1, p_2, \ldots, p_n) = (1, 1, \ldots, 1)$ on $\mathcal{F}_n$ is coherent.*

Definition 4

A p-consistent family $\mathcal{F}_n = \{E_i|H_i\,,\ i = 1,\ldots,n\}$ p-entails *a conditional event $E|H$ (denoted by $\mathcal{F}_n \Rightarrow_p E|H$) if and only if for any coherent probability assessment $(p_1, \ldots, p_n, z)$ on $\mathcal{F}_n \cup \{E|H\}$ it holds that: if $p_1 = \cdots = p_n = 1$, then $z = 1$.*

Of course, when $\mathcal{F}_n$ p-entails $E|H$, there may be coherent assessments $(p_1,\dots,p_n,z)$ with $z \neq 1$, but in such cases $p_i \neq 1$ for at least one index i. We say that the inference from a p-consistent family $\mathcal{F}_n$ to $E|H$ is *p-valid* if and only if $\mathcal{F}_n$ p-entails $E|H$.

We also recall the characterization of the p-entailment for two conditional events (Gilio and Sanfilippo 2013d, Theorem 7):

Theorem 2
Given two conditional events $A|H$, $B|K$, with $AH \neq \emptyset$. It holds that

$$A|H \Rightarrow_p B|K \iff A|H \subseteq B|K, \text{ or } K \subseteq B \iff \Pi \subseteq \{(x,y) \in [0,1]^2 : x \leq y\},$$

where Π is the set of coherent assessments (x,y) on $\{A|H, B|K\}$.

2.3 Conjunction, Disjunction, and Iterated Conditioning

Given two conditional events $A|H$ and $B|K$, the associated constituents, denoted $C_1,\dots,C_8,C_0$ in Table 1, are the conjunctions of the logical disjunction in the formula below.

$$\Omega = (AH \vee \bar{A}H \vee \bar{H}) \wedge (BK \vee \bar{B}K \vee \bar{K}) = AHBK \vee AH\bar{B}K \vee \cdots \vee \bar{H}\bar{K}.$$

	C_h	$A\|H$	$B\|K$	$\max\{A\|H + B\|K - 1, 0\}$	$(A\|H) \wedge (B\|K)$	$\min\{A\|H, B\|K\}$
C_1	$AHBK$	1	1	1	1	1
C_2	$AH\bar{B}K$	1	0	0	0	0
C_3	$AH\bar{K}$	1	y	y	y	y
C_4	$\bar{A}HBK$	0	1	0	0	0
C_5	$\bar{A}H\bar{B}K$	0	0	0	0	0
C_6	$\bar{A}H\bar{K}$	0	y	0	0	0
C_7	$\bar{H}BK$	x	1	x	x	x
C_8	$\bar{H}\bar{B}K$	x	0	0	0	0
C_0	$\bar{H}\bar{K}$	x	y	$\max\{x+y-1, 0\}$	z	$\min\{x,y\}$

Table 1: *Possible values of* $\max\{A|H + B|K - 1, 0\}$, $(A|H) \wedge (B|K)$, *and* $\min\{A|H, B|K\}$, *associated with the constituents* $C_1,\dots,C_8,C_0$, *where* $x = P(A|H)$, $y = P(B|K)$, *and* $z = \mathbb{P}[(A|H) \wedge (B|K)]$.

We recall now the notion of conjoined conditionals which was introduced in the framework of conditional random quantities (Gilio and Sanfilippo 2013b; Gilio and Sanfilippo 2013a; Gilio and Sanfilippo 2014; Gilio and Sanfilippo 2019). Given a coherent probability assessment (x,y) on $\{A|H, B|K\}$, we consider the random quantity $AHBK + x\bar{H}BK + y\bar{K}AH$ and we set $\mathbb{P}[(AHBK + x\bar{H}BK + y\bar{K}AH)|(H \vee K)] = z$. Then we define the conjunction $(A|H) \wedge (B|K)$ as follows:

Definition 5
Given a coherent prevision assessment $P(A|H) = x$, $P(B|K) = y$, and $\mathbb{P}[(AHBK + x\bar{H}BK + y\bar{K}AH)|(H \vee K)] = z$, the conjunction $(A|H) \wedge (B|K)$

is the conditional random quantity defined as

$$(A|H) \wedge (B|K) = (AHBK + x\overline{H}BK + y\overline{K}AH)|(H \vee K) = \\ = \begin{cases} 1, & \text{if } AHBK \text{ is true,} \\ 0, & \text{if } \overline{A}H \vee \overline{B}K \text{ is true,} \\ x, & \text{if } \overline{H}BK \text{ is true,} \\ y, & \text{if } AH\overline{K} \text{ is true,} \\ z, & \text{if } \overline{H}\,\overline{K} \text{ is true.} \end{cases} \tag{3}$$

Of course, $\mathbb{P}[(A|H) \wedge (B|K)] = z$. Notice that, once the (coherent) assessment (x, y, z) is given, the conjunction $(A|H) \wedge (B|K)$ is (subjectively) determined. We recall that, in betting terms, z represents the amount you agree to pay, with the proviso that you will receive the quantity

$$(A|H) \wedge (B|K) = AHBK + x\overline{H}BK + y\overline{K}AH + z\overline{H}\,\overline{K}, \tag{4}$$

which assumes one of the following values:

- 1, if both conditional events are true;
- 0, if at least one of the conditional events is false;
- the probability of the conditional event that is void, if one conditional event is void and the other one is true;
- z (the amount that you paid), if both conditional events are void.

We observe that $(A|H) \wedge (A|H) = A|H$ and $(A|H) \wedge (B|K) = (B|K) \wedge (A|H)$. Moreover, if $H = K$, then

$$(A|H) \wedge (B|H) = AB|H. \tag{5}$$

Indeed, in this case $\overline{H}BK = AH\overline{K} = \emptyset$, so that by Definition 5 it holds that $z = \mathbb{P}(ABH|H) = P(AB|H)$ and $(A|H) \wedge (B|H) = ABH|H = AB|H$. The result below shows that Fréchet-Hoeffding bounds still hold for the conjunction of two conditional events (Gilio and Sanfilippo 2014, Theorem 7).

Theorem 3

Given any coherent assessment (x, y) on $\{A|H, B|K\}$, with A, H, B, K logically independent, and with $H \neq \emptyset, K \neq \emptyset$, the extension $z = \mathbb{P}[(A|H) \wedge (B|K)]$ is coherent if and only if the following Fréchet-Hoeffding bounds are satisfied:

$$\max\{x + y - 1, 0\} = z' \leq z \leq z'' = \min\{x, y\}. \tag{6}$$

Remark 2

Notice that, from (3) and (6), it holds that (see Table 1)

$$\max\{A|H + B|K - 1, 0\} \leq (A|H) \wedge (B|K) \leq \min\{A|H, B|K\}. \tag{7}$$

Then, when $AH = \emptyset$, it holds that $A|H = 0$ and $(A|H) \wedge (B|K) = 0 \wedge (B|K) = 0$. Moreover, when $K \subseteq B$, it holds that $B|K = 1$ and $(A|H) \wedge (B|K) = (A|H) \wedge 1 = A|H$.

We recall now the notion of disjoined conditional. Given a coherent probability assessment (x,y) on $\{A|H,B|K\}$ we consider the random quantity $(AH\vee BK)+x\overline{H}\overline{B}K+y\overline{K}\overline{A}H$ and we set $\mathbb{P}[((AH\vee BK)+x\overline{H}\overline{B}K+y\overline{K}\overline{A}H)|(H\vee K)] = w$. Then we define the disjunction $(A|H)\vee(B|K)$ as follows:

Definition 6

Given a coherent prevision assessment $P(A|H) = x$, $P(B|K) = y$, and $\mathbb{P}[((AH \vee BK) + x\overline{H}\overline{B}K + y\overline{K}\overline{A}H)|(H \vee K)] = w$, the disjunction $(A|H) \vee (B|K)$ is the conditional random quantity defined as

$$(A|H) \vee (B|K) = ((AH \vee BK) + x\overline{H}\overline{B}K + y\overline{K}\overline{A}H)|(H \vee K) = \\ = \begin{cases} 1, & \text{if } AH \vee BK \text{ is true,} \\ 0, & \text{if } \overline{A}H\overline{B}K \text{ is true,} \\ x, & \text{if } \overline{H}\overline{B}K \text{ is true,} \\ y, & \text{if } \overline{A}H\overline{K} \text{ is true,} \\ w, & \text{if } \overline{H}\overline{K} \text{ is true.} \end{cases} \tag{8}$$

Remark 3

Given any coherent assessment (x,y) on $\{A|H,B|K\}$, with A,H,B,K logically independent, and with $H \neq \emptyset, K \neq \emptyset$, the extension $w = \mathbb{P}[(A|H) \vee (B|K)]$ is coherent if and only if (Gilio and Sanfilippo 2014, Section 6)

$$\max\{x,y\} = w' \leq w \leq w'' = \min\{1,x+y\}. \tag{9}$$

Notice that, from (8) and (9), it holds that

$$\max\{A|H,B|K\} \leq (A|H) \vee (B|K) \leq \min\{1,A|H+B|K\}. \tag{10}$$

Then, when $AH = \emptyset$, it holds that $A|H = 0$ and $(A|H) \vee (B|K) = 0 \vee (B|K) = B|K$. Moreover, when $K \subseteq B$, it holds that $B|K = 1$ and $(A|H) \vee (B|K) = (A|H) \vee 1 = 1$.

We recall that, as defined in (1), the indicator of a conditional event $A|H$, with $P(A|H) = x$, is

$$A|H = A \wedge H + x\overline{H}.$$

Likewise, we define the notion of an iterated conditional based on the same structure, i.e. $\square|\bigcirc = \square\wedge\bigcirc+\mathbb{P}(\square|\bigcirc)\overline{\bigcirc}$, where $\square$ denotes $B|K$ and $\bigcirc$ denotes $A|H$, and where we set $\mathbb{P}(\square|\bigcirc) = \mu$. In the framework of subjective probability $\mu = \mathbb{P}(\square|\bigcirc)$ is the amount that you agree to pay, by knowing that you will receive the random quantity $\square\wedge\bigcirc+\mu\overline{\bigcirc}$. The negation $\overline{A|H}$ of $A|H$ is defined as $1-A|H = \overline{A}|H$. Then, the iterated conditional $(B|K)|(A|H)$ is defined (see, e.g., Gilio and Sanfilippo 2013b; Gilio and Sanfilippo 2013a; Gilio and Sanfilippo 2014) as follows:

Definition 7

Given any pair of conditional events $A|H$ and $B|K$, with $AH \neq \emptyset$, let (x,y,z) be a coherent assessment on $\{A|H,B|K,(A|H) \wedge (B|K)\}$. The iterated conditional

$(B|K)|(A|H)$ *is defined as*

$$(B|K)|(A|H) = (B|K) \wedge (A|H) + \mu \bar{A}|H =$$

$$\begin{cases} 1, & \text{if } AHBK \text{ is true,} \\ 0, & \text{if } AH\bar{B}K \text{ is true,} \\ y, & \text{if } AH\bar{K} \text{ is true,} \\ x + \mu(1-x), & \text{if } \bar{H}BK \text{ is true,} \\ \mu(1-x), & \text{if } \bar{H}\bar{B}K \text{ is true,} \\ z + \mu(1-x), & \text{if } \bar{H}\bar{K} \text{ is true,} \\ \mu, & \text{if } \bar{A}H \text{ is true,} \end{cases} \tag{11}$$

where $\mu = \mathbb{P}[(B|K)|(A|H)] = \mathbb{P}[(B|K) \wedge (A|H) + \mu \bar{A}|H]$.

Notice that we assume $AH \neq \emptyset$ to avoid trivial cases of iterated conditionals. By the linearity of prevision, it holds that

$$\begin{aligned} \mu &= \mathbb{P}((B|K)|(A|H)) = \mathbb{P}((B|K) \wedge (A|H) + \mu \bar{A}|H) = \\ &= \mathbb{P}((B|K) \wedge (A|H)) + \mathbb{P}(\mu \bar{A}|H) = \\ &= \mathbb{P}((B|K) \wedge (A|H)) + \mu P(\bar{A}|H) = z + \mu(1-x), \end{aligned}$$

from which it follows that (Gilio and Sanfilippo 2013a)

$$z = \mathbb{P}((B|K) \wedge (A|H)) = \mu x = \mathbb{P}((B|K)|(A|H))P(A|H), \tag{12}$$

and $\mu = \mathbb{P}((B|K)|(A|H)) = \frac{\mathbb{P}((B|K) \wedge (A|H))}{P(A|H)} = \frac{z}{x} \in [0,1]$, when $x > 0$. We observe that, when $x = 0$, one has

$$(B|K)|(A|H) = \begin{cases} 1, & \text{if } AHBK \text{ is true,} \\ 0, & \text{if } AH\bar{B}K \text{ is true,} \\ y, & \text{if } AH\bar{K} \text{ is true,} \\ \mu, & \text{if } \bar{A}H \vee \bar{H} \text{ is true.} \end{cases}$$

Then, in order that the prevision assessment μ on $(B|K)|(A|H)$ be coherent, μ must belong to the convex hull of the values $0, y, 1$; that is, (also when $x = 0$) it must be that $\mu \in [0,1]$. Therefore in all cases $(B|K)|(A|H) \in [0,1]$.

The notions of conjunction $\mathcal{C}_{1\cdots n} = (E_1|H_1) \wedge \cdots \wedge (E_n|H_n)$ and disjunction $\mathcal{D}_{1\cdots n} = (E_1|H_1) \vee \cdots \vee (E_n|H_n)$ of n conditional events have been defined as (Gilio and Sanfilippo 2019, see also Gilio and Sanfilippo 2020)

$$\mathcal{C}_{1\cdots n} = \begin{cases} 1, & \text{if } \bigwedge_{i=1}^n E_i H_i, \text{ is true} \\ 0, & \text{if } \bigvee_{i=1}^n \bar{E}_i H_i, \text{ is true,} \\ x_S, & \text{if } \bigwedge_{i \in S} \bar{H}_i \bigwedge_{i \notin S} E_i H_i \text{ is true,} \end{cases} \tag{13}$$

and

$$\mathcal{D}_{1\cdots n} = \begin{cases} 1, & \text{if } \bigvee_{i=1}^n E_i H_i, \text{ is true} \\ 0, & \text{if } \bigwedge_{i=1}^n \bar{E}_i H_i, \text{ is true,} \\ y_S, & \text{if } \bigwedge_{i \in S} \bar{H}_i \bigwedge_{i \notin S} \bar{E}_i H_i \text{ is true,} \end{cases} \tag{14}$$

where, for each non-empty subset S of $\{1,\dots,n\}$, x_S is the prevision of $\bigwedge_{i\in S}(E_i|H_i)$ and y_S is the prevision of $\bigvee_{i\in S}(E_i|H_i)$. Notice that $\mathcal{C}_{1\cdots n}$ and $\mathcal{D}_{1\cdots n}$ are conditional random quantities, with conditioning event $\bigvee_{i=1}^{n} H_i$ and with a finite set of possible values in $[0,1]$.

3. A Survey of Basic Properties, from Unconditional to Conditional Events

In this section we recall some well known logical and probabilistic properties for the case of unconditional events. Then, we illustrate analogous properties for the case of conditional events.

3.1 Some Basic Properties of Unconditional Events

We recall below some basic properties which concern unconditional events. The indicator of an event E is a random quantity, denoted by the same symbol, which is 1, or 0, according to whether E is true, or false, respectively.

1. Given two events A and B, denoting by the same symbols their indicators, it holds that
$$A \leq B \Longleftrightarrow A \subseteq B \iff AB = A,\ A \vee B = B\,.$$
2. Under the hypothesis $\emptyset \neq A \subseteq B$, one has $P(B|A) = 1$ and $B|A = AB + P(B|A)\overline{A} = A + \overline{A} = 1$.
3. Logical and probabilistic relations between disjunction and conjunction:
$$A \vee B = A + B - AB, \quad P(A \vee B) = P(A) + P(B) - P(AB).$$
4. De Morgan's laws
$$\overline{AB} = \overline{A} \vee \overline{B}, \quad \overline{A \vee B} = \overline{A}\,\overline{B}.$$
5. Inclusion-exclusion principle
$$E_1 \vee \cdots \vee E_n = \sum_{i=1}^{n} E_i - \sum_{i<j} E_iE_j + \cdots + (-1)^{n+1} E_1 \cdots E_n.$$
6. Fréchet-Hoeffding bounds
$$\max\{\sum_{i=1}^{n} P(E_i) - n + 1, 0\} \leq P(E_1 \cdots E_n) \leq \min\{P(E_1),\dots,P(E_n)\}.$$
7. Probability consistency. A family $\mathcal{F} = \{E_1,\dots,E_n\}$ of n events is p-consistent if the assessment $P(E_i) = 1$, $i = 1,\dots,n$, is coherent. We observe that
$$P(E_i) = 1, i = 1,\dots,n \iff P(E_1 \cdots E_n) = 1\,.$$
Indeed, defining $P(E_i) = x_i, i = 1,\dots,n$, and $P(E_1 \cdots E_n) = z$, by the Fréchet-Hoeffding bounds it holds that
$$\max\{x_1 + \cdots + x_n - n + 1, 0\} \leq z \leq \min\{x_1,\dots,x_n\},$$
from which it follows that $x_i = 1, i = 1,\dots,n,$ if and only if $z = 1$. Then, p-consistency amounts to the coherence of the assessment $P(E_1 \cdots E_n) = 1$.

8. Probabilistic entailment. Given a p-consistent family $\mathcal{F} = \{E_1, \dots, E_n\}$ and a further event E_{n+1}, the family $\mathcal{F}$ p-entails the event E_{n+1} if and only if $P(E_i) = 1, i = 1, \dots, n$, implies that $P(E_{n+1}) = 1$. The p-entailment of E_{n+1} from $\mathcal{F}$ is equivalent to each one of the following properties

   (i) $E_1 \cdots E_n \subseteq E_{n+1}$, that is $E_1 \cdots E_n E_{n+1} = E_1 \cdots E_n$

   (ii) the (indicator of the) conditional event $E_{n+1}|E_1 \cdots E_n$ is constant and coincides with 1.

   We first observe that, given two events A and B, it holds that

   $$A \text{ p-entails } B \iff A \subseteq B, \text{ that is } A \leq B.$$

   Indeed, if $A \subseteq B$, then $P(A) = 1$ implies $P(B) = 1$. Conversely, by assuming that A p-entails B, if it were $A \nsubseteq B$, i.e. $A\bar{B} \neq \emptyset$, then given any assessment

   $$P(AB) = p\,,\; P(A\bar{B}) = 1 - p\,, \text{ with } p < 1\,,$$

   it would follow $P(A) = 1$, $P(\bar{A}B) = 0$, and $P(B) = P(AB) + P(\bar{A}B) = p < 1$, which contradicts the hypothesis. Then, the p-entailment of the event E_{n+1} from the event $E_1 \cdots E_n$ amounts to $E_1 \cdots E_n \subseteq E_{n+1}$, that is $E_1 \cdots E_n \leq E_{n+1}$. Then, it can be verified that

   $$\mathcal{F} = \{E_1, \dots, E_n\} \text{ p-entails } E_{n+1} \iff E_1 \cdots E_n \text{ p-entails } E_{n+1}\,.$$

   Indeed, as $P(E_1 \cdots E_n) = 1$ is equivalent to $P(E_i) = 1, i = 1, \dots, n$, if $\mathcal{F}$ p-entails E_{n+1}, then $P(E_1 \cdots E_n) = 1$ implies $P(E_{n+1}) = 1$, that is $E_1 \cdots E_n$ p-entails E_{n+1}. Conversely, as $P(E_i) = 1, i = 1, \dots, n$, is equivalent to $P(E_1 \cdots E_n) = 1$, if $E_1 \cdots E_n$ p-entails E_{n+1}, then $P(E_i) = 1, i = 1, \dots, n$, implies $P(E_{n+1}) = 1$, that is $\mathcal{F}$ p-entails E_{n+1}. Therefore

   $$\mathcal{F} \text{ p-entails } E_{n+1} \iff E_1 \cdots E_n \text{ p-entails } E_{n+1} \iff E_1 \cdots E_n \leq E_{n+1}\,,$$

   that is, the p-entailment of E_{n+1} from $\mathcal{F}$ is equivalent to the property (i).

   Concerning the property (ii), if $\mathcal{F}$ p-entails E_{n+1}, then $E_1 \cdots E_n \subseteq E_{n+1}$ and hence, from (2), $E_{n+1}|E_1 \cdots E_n$ is constant and coincides with 1.

   Conversely, if $E_{n+1}|E_1 \cdots E_n$ coincides with 1, then $P(E_{n+1}|E_1 \cdots E_n) = 1$ and $E_1 \cdots E_n \overline{E_{n+1}} = \emptyset$, that is $E_1 \cdots E_n \subseteq E_{n+1}$, from which it follows that $E_1 \cdots E_n$ p-entails E_{n+1} and hence $\mathcal{F}$ p-entails E_{n+1}. Therefore, the p-entailment of E_{n+1} from $\mathcal{F}$ is equivalent to the property (ii).

3.2 Basic Properties of Conditional Events

In this section we show that the basic properties considered in Section 3.1 continue to hold when replacing events by conditional events. Here we denote by h^* the property h in the previous section.

1.* Given two conditional events $A|H$ and $B|K$, it holds that

   $$A|H \leq B|K \iff A|H \subseteq B|K, \text{ or } AH = \emptyset, \text{ or } K \subseteq B, \tag{15}$$

   and

   $$A|H \leq B|K \iff (A|H) \wedge (B|K) = A|H\,,\; (A|H) \vee (B|K) = B|K. \tag{16}$$

   Indeed, concerning (15), if $AH = \emptyset$, or $K \subseteq B$, then $A|H = 0$, or $B|K = 1$, and trivially it holds that $A|H \leq B|K$. If $A|H \subseteq B|K$, then by coherence

$P(A|H) \leq P(B|K)$ and hence $A|H \leq B|K$. Conversely, if $A|H \leq B|K$, then $P(A|H) \leq P(B|K)$, and the inequality may be satisfied, in three different ways, because: $AH = \emptyset$ (in which case $A|H = 0 \leq B|K$), or $K \subseteq B$ (in which case $B|K = 1 \geq A|H$), or $A|H \subseteq B|K$. For more details, see Gilio and Sanfilippo 2013d, Theorem 6. Moreover, concerning (16), if $A|H \leq B|K$, then by (15), we have three cases: (a) $AH = \emptyset$; (b) $K \subseteq B$; (c) $A|H \subseteq B|K$.

Case (a). It holds that $A|H = 0$ and by Remarks 2 and 3 it follows that

$$(A|H) \wedge (B|K) = 0 \wedge (B|K) = 0 = A|H, \ \ (A|H) \vee (B|K) = 0 \vee (B|K) = B|K.$$

Case (b). It holds that $B|K = 1$ and by Remarks 2 and 3 it follows that

$$(A|H) \wedge (B|K) = (A|H) \wedge 1 = A|H, \ \ (A|H) \vee (B|K) = (A|H) \vee 1 = 1 = B|K.$$

Case (c). If $A|H \subseteq B|K$, that is $AH \subseteq BK$ and $\bar{B}K \subseteq \bar{A}H$, it holds that $AH\bar{B}K = AH\bar{K} = \bar{H}\bar{B}K = \emptyset$ and the constituents are $C_1 = AHBK, C_2 = \bar{A}HBK, C_3 = \bar{A}H\bar{B}K, C_4 = \bar{A}H\bar{K}, C_5 = \bar{H}BK, C_0 = \bar{H}\bar{K}$. By defining $P(A|H) = x$, $P(B|K) = y$, $z = \mathbb{P}[(A|H) \wedge (B|K)]$, and $w = \mathbb{P}[(A|H) \vee (B|K)]$, the possible values of $A|H$, $B|K$, $(A|H) \wedge (B|K)$ and $(A|H) \vee (B|K)$ are illustrated in Table 2.

From Table 2, we observe that $(A|H) \wedge (B|K) = A|H$ when $H \vee K$ is true. Then, by Theorem 1 it follows that $z = x$; therefore $(A|H) \wedge (B|K) = A|H$ in all cases (see also Gilio and Sanfilippo 2013a, Section 3). Likewise, we observe that $(A|H) \vee (B|K) = B|K$ when $H \vee K$ is true. Then, by Theorem 1 it follows that $w = y$; therefore $(A|H) \vee (B|K) = B|K$ in all cases.

	C_h	$A\|H$	$B\|K$	$(A\|H) \wedge (B\|K)$	$(A\|H) \vee (B\|K)$
C_1	$AHBK$	1	1	1	1
C_2	$\bar{A}HBK$	0	1	0	1
C_3	$\bar{A}H\bar{B}K$	0	0	0	0
C_4	$\bar{A}H\bar{K}$	0	y	0	y
C_5	$\bar{H}BK$	x	1	x	1
C_0	$\bar{H}\bar{K}$	x	y	z	w

Table 2: *Possible values of* $A|H$, $B|K$, $(A|H) \wedge (B|K)$ *and* $(A|H) \vee (B|K)$, *when* $A|H \subseteq B|K$.

2.* We show the analogous of property 2 in terms of iterated conditionals. Under the hypothesis $0 \neq A|H \leq B|K$, that is $K \subseteq B$, or $AH \neq \emptyset$ and $A|H \subseteq B|K$, it can be verified that $\mathbb{P}[(B|K)|(A|H)] = 1$ and $(B|K)|(A|H) = 1$. Indeed, defining $\mathbb{P}[(B|K)|(A|H)] = \mu$, from (16) it holds that $(B|K) \wedge (A|H) = A|H$; then

$$(B|K)|(A|H) = A|H + \mu\bar{A}|H = \begin{cases} 1, & \text{if } AH \text{ is true,} \\ \mu, & \text{if } \bar{A}H \text{ is true,} \\ x + \mu(1-x), & \text{if } \bar{H} \text{ is true.} \end{cases}$$

By linearity of prevision it holds that

$$\mu = x + \mu(1-x);$$

then $(B|K)|(A|H) \in \{1, \mu\}$ and, by coherence, $\mu = 1$. For a discussion of this result in the context of connexive logic see Pfeifer and Sanfilippo 2021.

3.* Relation between disjunction and conjunction of conditional events (Gilio and Sanfilippo 2014, Section 6)

$$\mathbb{P}[(A|H) \vee (B|K)] = P(A|H) + P(B|K) - \mathbb{P}[(A|H) \wedge (B|K)],$$

and

$$(A|H) \vee (B|K) = A|H + B|K - (A|H) \wedge (B|K).$$

4.* De Morgan's laws for conjunction and disjunction of conditional events (Gilio and Sanfilippo 2019, Theorem 5).

$$\overline{(A|H) \wedge (B|K)} = (\bar{A}|H) \vee (\bar{B}|K), \quad \overline{(A|H) \vee (B|K)} = (\bar{A}|H) \wedge (\bar{B}|K),$$

where

$$\overline{(A|H) \wedge (B|K)} = 1 - (A|H) \wedge (B|K),$$

and

$$\overline{(A|H) \vee (B|K)} = 1 - (A|H) \vee (B|K).$$

5.* Inclusion-exclusion principle for conditional events.

Given n conditional events $E_1|H_1, \dots, E_n|H_n$, by recalling (13) and (14), it holds that (Gilio and Sanfilippo 2020)

$$\mathcal{D}_{1\cdots n} = \sum_{i=1}^{n} \mathcal{C}_i - \sum_{1 \le i_1 < i_2 \le n} \mathcal{C}_{i_1 i_2} + \cdots + (-1)^{n+1} \mathcal{C}_{1\cdots n},$$

where $\mathcal{C}_{i_1\cdots i_k} = (E_{i_1}|H_{i_1}) \wedge \cdots \wedge (E_{i_k}|H_{i_k})$, $\{i_1, \dots, i_k\} \subseteq \{1, \dots, n\}$.

6.* Fréchet-Hoeffding bounds for the conjunction of conditional events. Given a family of n conditional events $\mathcal{F} = \{E_1|H_1, \dots, E_n|H_n\}$, let $\mathcal{P} = (x_1, \dots, x_n)$ be a coherent probability assessment on $\mathcal{F}$, with $x_i = P(E_i|H_i), i = 1, \dots, n$. We set $\mathbb{P}(\mathcal{C}_{1\cdots n}) = z$. Then, under logical independence of $E_1, \dots, E_n, H_1, \dots, H_n$, z is a coherent extension of $(x_1, , x_n)$ if and only if

$$\max\{x_1 + \cdots + x_n - n + 1, 0\} \le z \le \min\{x_1, \dots, x_n\}, \tag{17}$$

that is the Fréchet-Hoeffding bounds continue to hold for our notion of conjunction of conditional events. The necessity of (17) has been proved in Gilio and Sanfilippo 2019, while the sufficiency has been proved in Gilio and Sanfilippo 2021a.

7.* Probabilistic consistency of a family of conditional events. A family of n conditional events $\mathcal{F} = \{E_1|H_1, \dots, E_n|H_n\}$ is defined p-consistent if the assessment $P(E_1|H_1) = \cdots = P(E_n|H_n) = 1$ is coherent. It holds that (see Gilio and Sanfilippo 2019, proof of Theorem 17)

$$P(E_1|H_1) = \cdots = P(E_n|H_n) = 1 \Longleftrightarrow \mathbb{P}[(E_1|H_1) \wedge \cdots \wedge (E_n|H_n)] = 1.$$

Then,

$$\mathcal{F} \text{ is p-consistent} \Longleftrightarrow \mathbb{P}[(E_1|H_1) \wedge \cdots \wedge (E_n|H_n)] = 1 \text{ is coherent.}$$

8.* Probabilistic entailment from a family of conditional events. Given a p-consistent family of n conditional events $\mathcal{F} = \{E_1|H_1, \dots, E_n|H_n\}$ and a further conditional event $E_{n+1}|H_{n+1}$, we say that $\mathcal{F}$ p-entails $E_{n+1}|H_{n+1}$ if and only if $P(E_i|H_i) = 1, i = 1, \dots, n$, implies $P(E_{n+1}|H_{n+1}) = 1$. It can be verified (Gilio and Sanfilippo 2019, Theorem 18) that the following three properties are equivalent:

(a) $\mathcal{F}$ p-entails $E_{n+1}|H_{n+1}$;

(b) $(E_1|H_1) \wedge \cdots \wedge (E_n|H_n) \leq E_{n+1}|H_{n+1}$;

(c) $(E_1|H_1) \wedge \cdots \wedge (E_n|H_n) \wedge E_{n+1}|H_{n+1} = (E_1|H_1) \wedge \cdots \wedge (E_n|H_n)$.

In particular, when $n = 1$, a p-consistent conditional event $E_1|H_1$ p-entails $E_2|H_2$ if and only if $(E_1|H_1) \leq (E_2|H_2)$, that is (as shown by property 2*; see also Gilio, Pfeifer and Sanfilippo 2020, Theorem 4)

$$E_1|H_1 \text{ p-entails } E_2|H_2 \iff (E_2|H_2)|(E_1|H_1) = 1, \tag{18}$$

where $E_1H_1 \neq \emptyset$. In Gilio and Sanfilippo 2019, Definition 14, the notion of iterated conditional $(E_{n+1}|H_{n+1})|((E_1|H_1) \wedge \cdots \wedge (E_n|H_n))$, with $(E_1|H_1) \wedge \cdots \wedge (E_n|H_n) \neq 0$, has been defined as the following random quantity

$$(E_1|H_1) \wedge \cdots \wedge (E_{n+1}|H_{n+1}) + \mu(1 - (E_1|H_1) \wedge \cdots \wedge (E_n|H_n)),$$

where $\mu = \mathbb{P}[(E_{n+1}|H_{n+1})|((E_1|H_1) \wedge \cdots \wedge (E_n|H_n))]$. In particular,

$$\begin{aligned}&(E_3|H_3)|((E_1|H_1) \wedge (E_2|H_2)) = (E_1|H_1) \wedge (E_2|H_2) \wedge (E_3|H_3)+\\ &+\mu(1 - (E_1|H_1) \wedge (E_2|H_2)),\end{aligned}$$

where $\mu = \mathbb{P}[(E_3|H_3)|((E_1|H_1) \wedge (E_2|H_2))]$ and

$$(E_1|H_1)\wedge(E_2|H_2)\wedge(E_3|H_3) = \begin{cases} 1, & \text{if } E_1H_1E_2H_2E_3H_3 \text{ is true,}\\ 0, & \text{if } \overline{E}_1H_1 \vee \overline{E}_2H_2 \vee \overline{E}_3H_3 \text{ is true,}\\ x_1, & \text{if } \overline{H}_1E_2H_2E_3H_3 \text{ is true,}\\ x_2, & \text{if } E_1H_1\overline{H}_2E_3H_3 \text{ is true,}\\ x_3, & \text{if } E_1H_1E_2H_2\overline{H}_3 \text{ is true,}\\ x_{12}, & \text{if } \overline{H}_1\overline{H}_2E_3H_3 \text{ is true,}\\ x_{13}, & \text{if } \overline{H}_1E_2H_2\overline{H}_3 \text{ is true,}\\ x_{23}, & \text{if } E_1H_1\overline{H}_2\overline{H}_3 \text{ is true,}\\ x_{123}, & \text{if } \overline{H}_1\overline{H}_2\overline{H}_3 \text{ is true,}\end{cases} \tag{19}$$

where $x_i = P(E_i|H_i)$, $i = 1, 2, 3$, $x_{ij} = x_{ji} = \mathbb{P}[(E_i|H_i) \wedge (E_j|H_j)]$, $i \neq j$, and $x_{123} = \mathbb{P}[(E_1|H_1)\wedge(E_2|H_2)\wedge(E_3|H_3)]$. In Gilio, Pfeifer and Sanfilippo 2020, given a p-consistent family $\mathcal{F} = \{E_1|H_1, E_2|H_2\}$ and a further event $E_3|H_3$, it has been proved that the p-entailment of $E_3|H_3$ from $\mathcal{F}$ is equivalent to the property that the iterated conditional $(E_3|H_3)|((E_1|H_1)\wedge(E_2|H_2))$ is constant and equal to 1, that is

$$\begin{aligned}&\{E_1|H_1, E_2|H_2\} \text{ p-entails } E_3|H_3 \iff\\ &(E_3|H_3)|((E_1|H_1) \wedge (E_2|H_2)) = 1,\end{aligned} \tag{20}$$

where $\{E_1|H_1, E_2|H_2\}$ is p-consistent. For the extension of (20) to the general case see Gilio and Sanfilippo 2021b.

4. On Iterated Conditionals

In this section we deepen some aspects and applications of iterated conditionals. In the next subsection we show that some complex sentences on conditionals, which seem intuitively acceptable, can be analyzed in a rigorous way in terms of iterated conditionals.

4.1 Complex Sentences and Iterated Conditionals

Given an indicative conditional "*if H then A*", simply denoted $\mathcal{C}$, let us consider the following intuitively valid assertions:

(a) the probability of $\mathcal{C}$ is (not the probability of its truth, but) the probability of its truth, given that it is true or false;
(b) the probability of $\mathcal{C}$, given that A and H are true, is 1;
(c) the probability of $\mathcal{C}$, given that A is false and H is true, is 0;
(d) the probability of $\mathcal{C}$, given that H is false, is $P(A|H)$;
(e) the probability of $\mathcal{C}$, given that H is true, is $P(A|H)$;
(f) the probability of $\mathcal{C}$, given that "if H then A", is 1;
(g) the probability of $\mathcal{C}$, given that "if H then $\bar{A}$", is 0;
(h) is it the case that the probability of $\mathcal{C}$, given that $\overline{AH}$, is equal to 0?

We show below that the previous assertions have a clear meaning in the context of iterated conditionals.

(a) We consider the compound conditional "if $\mathcal{C}$ is true or false, then $\mathcal{C}$ is true", which can be directly represented by the conditional event $AH|(AH \vee \bar{A}H) = AH|H = A|H$, so that the probability of $\mathcal{C}$ is $P(AH|(AH \vee \bar{A}H)) = P(A|H)$. Then, the probability of $\mathcal{C}$ is the probability of its truth, given that it is true or false.
(b) We consider the compound conditional "if AH then $\mathcal{C}$" and we represent it by the iterated conditional $(A|H)|AH$. We observe that $AH \subseteq A|H$ and hence $(A|H) \wedge AH = AH$. Then,

$$(A|H)|(AH) = (A|H) \wedge AH + \mu\overline{AH} = AH + \mu\overline{AH},$$

which is equal to 1, or μ, according to whether AH is true, or false, respectively. By coherence $\mu = 1$ and hence $(A|H)|(AH) = 1$. The same result follows by exploiting the representation $A|H = AH + x\bar{H}$, where $x = P(A|H)$. We observe that $\bar{H}|AH = 0$; then

$$(A|H)|AH = (AH + x\bar{H})|AH = AH|AH = 1,$$

and hence $\mathbb{P}[(A|H)|AH] = P(AH|AH) = 1$. Thus, the conditional "if AH then $\mathcal{C}$" is the iterated conditional $(A|H)|AH$, which coincides with the constant $AH|AH = 1$ and has probability 1.
(c) We consider the compound conditional "if $\bar{A}H$ then $\mathcal{C}$" and represent it by the iterated conditional $(A|H)|\bar{A}H$. We set $P(A|H) = x$ and we observe that $AH|\bar{A}H = \bar{H}|\bar{A}H = 0$; then

$$(A|H)|\bar{A}H = (AH + x\bar{H})|\bar{A}H = AH|\bar{A}H + x\bar{H}|\bar{A}H = 0,$$

and hence $\mathbb{P}[(A|H)|\bar{A}H] = 0$. Thus, the conditional "if $\bar{A}H$ then $\mathcal{C}$" is the iterated conditional $(A|H)|\bar{A}H$, which coincides with the constant 0 and has probability 0.

(d) We consider the compound conditional "if $\overline{H}$ then $\mathcal{C}$" and represent it by the iterated conditional $(A|H)|\overline{H}$. We set $P(A|H) = x$ and we observe that $AH|\overline{H} = 0$ and $\overline{H}|\overline{H} = 1$; then

$$(A|H)|\overline{H} = (AH + x\overline{H})|\overline{H} = x\overline{H}|\overline{H} = x,$$

and hence $\mathbb{P}[(A|H)|\overline{H}] = x = P(A|H)$. Thus, the conditional "if $\overline{H}$ then $\mathcal{C}$" is the iterated conditional $(A|H)|\overline{H}$, which coincides with the constant x and has probability $x = P(A|H)$.

(e) We consider the compound conditional "if H then $\mathcal{C}$" and represent it by the iterated conditional $(A|H)|H$. Then, defining $P(A|H) = x$ and by observing that $\overline{H}|H = 0$, it holds that

$$(A|H)|H = (AH + x\overline{H})|H = A|H,$$

and hence $\mathbb{P}[(A|H)|H] = P(A|H)$. Thus, the conditional "if H then $\mathcal{C}$" is the iterated conditional $(A|H)|H$, which coincides with $A|H$ and its probability is $P(A|H)$. We observe that the conditional "if H then $\mathcal{C}$" is equivalent to the conditional "if $\mathcal{C}$ is true or false, then $\mathcal{C}$".

(f) We consider the compound conditional "if $\mathcal{C}$ then $\mathcal{C}$" and we represent it by the iterated conditional $(A|H)|(A|H)$. We set $\mathbb{P}[(A|H)|(A|H)] = \mu$, $P(A|H) = x$ and we recall that $(A|H) \wedge (A|H) = A|H$. Then

$$(A|H)|(A|H) = (A|H) \wedge (A|H) + \mu\overline{A}|H = A|H + \mu\overline{A}|H = \begin{cases} 1, & \text{if } AH \text{ is true.} \\ \mu, & \text{if } \overline{A}H \text{ is true,} \\ x + \mu(1-x), & \text{if } \overline{H} \text{ is true.} \end{cases}$$

We observe that

$$\mu = \mathbb{P}[A|H + \mu(1 - A|H)] = P(A|H) + \mu(1 - P(A|H)) = x + \mu(1-x).$$

Then, $(A|H)|(A|H) \in \{1, \mu\}$ and by coherence it must be $\mu = 1$. Thus, the compound conditional "if $\mathcal{C}$ then $\mathcal{C}$ is the iterated conditional $(A|H)|(A|H)$, which is the constant 1 and has probability 1.

(g) We consider the compound conditional "if (if H then $\overline{A}$), then $\mathcal{C}$" and we represent it by the iterated conditional $(A|H)|(\overline{A}|H)$. We set $\mathbb{P}[(A|H)|(\overline{A}|H)] = \mu$, $P(A|H) = x$ and we observe that $(A|H) \wedge (\overline{A}|H) = 0$ and $\overline{\overline{A}|H} = 1 - \overline{A}|H = A|H$. Then

$$(A|H)|(\overline{A}|H) = (A|H) \wedge (\overline{A}|H) + \mu\overline{\overline{A}|H} = \mu A|H = \begin{cases} \mu, & \text{if } AH \text{ is true.} \\ 0, & \text{if } \overline{A}H \text{ is true,} \\ \mu x, & \text{if } \overline{H} \text{ is true,} \end{cases}$$

so that $\mu = \mathbb{P}[(A|H)|(\overline{A}|H)] = \mu P(A|H) = \mu x$. If $x < 1$, then $\mu = 0$. If $x = 1$, then $(A|H)|(\overline{A}|H) \in \{0, \mu\}$ and by coherence it must be $\mu = 0$. Thus, the compound conditional "if (if H then $\overline{A}$), then $\mathcal{C}$" is the iterated conditional $(A|H)|(\overline{A}|H)$, which is the constant 0 and has probability 0. Likewise, it holds that $(\overline{A}|H)|(A|H) = 0$.

(h) We consider the compound conditional "if $\overline{AH}$ then $\mathcal{C}$" and represent it by the iterated conditional $(A|H)|\overline{AH}$. Then, defining $P(A|H) = x$ and

$\mathbb{P}[(A|H)|\overline{AH}] = \mu$, by observing that $AH|\overline{AH} = 0$, we obtain

$$(A|H)|\overline{AH} = (AH + x\overline{H})|\overline{AH} = x\overline{H}|\overline{AH} = x\overline{H}|(\overline{A} \vee \overline{H}), \tag{21}$$

and hence $\mu = \mathbb{P}[(A|H)|\overline{AH}] = P(A|H)P[\overline{H}|(\overline{A}\vee\overline{H})] = xP[\overline{H}|(\overline{A}\vee\overline{H})]$, which in general is not 0.

Notice that in Edgington 2020, page 51, it is observed that in Bradley's theory (Bradley 2012) the probability of "$\mathcal{C}$, given $\overline{AH}$" is 0, instead of $P(A|H)P(\overline{H}|(\overline{A}\vee\overline{H})$. This is clearly not correct; indeed, to assume $\overline{AH}$ true amounts to assuming $\overline{A} \vee \overline{H}$ true, that is $\overline{A}H \vee \overline{H}$ true. Then, based on (22), in the betting framework, the conditional prevision $\mu = \mathbb{P}(\mathcal{C}|\overline{AH}) = \mathbb{P}[x\overline{H}|(\overline{A}H \vee \overline{H})]$ is the amount that should be paid in a conditional bet in order to receive 0 (when $\overline{A}H$ is true), or x (when $\overline{H}$ is true), with probability $P[\overline{A}H|(\overline{A}H \vee \overline{H})]$, or probability $P[\overline{H}|(\overline{A}H \vee \overline{H})]$, respectively (with the bet called off when AH is true). Hence, the amount μ to be paid is (not 0, but)

$$\mu = 0 \cdot P[\overline{A}H|(\overline{A}H \vee \overline{H})] + xP[\overline{H}|(\overline{A}H \vee \overline{H})] = xP[\overline{H}|(\overline{A}H \vee \overline{H})]. \tag{22}$$

4.2 Import-Export Principle

Given three events A, H, K, with $HK \neq \emptyset$, if the Import-Export principle (McGee 1989) were satisfied, then it would be $(A|H)|K = A|HK$. In our approach, in general, the Import-Export principle does not hold (Gilio and Sanfilippo 2014), that is $(A|H)|K \neq A|HK$; moreover $(A|H)|K \neq (A|K)|H$. The Import-Export principle holds when $H \subseteq K$, or $K \subseteq H$, in which case $(A|H)|K = (A|K)|H = A|HK$. We also observe that $A|(H|K) \neq A|HK$ (Sanfilippo, Gilio, Over et al. 2020, Remark 7). We illustrate by an example the non validity of the Import-Export principle. Let us consider the iterated conditional $(A|H)|(\overline{H}\vee A)$, where $(\overline{H}\vee A)$ is the material conditional associated to "if H then A". If the Import-Export principle were valid it would be

$$(A|H)|(\overline{H} \vee A) = (A|(H \wedge (\overline{H} \vee A)) = A|AH = 1.$$

On the contrary, defining $P(A|H) = x$ and $\mathbb{P}[(A|H)|(\overline{H} \vee A)] = \mu$, as $A|H \subseteq (\overline{H} \vee A)$ it holds that $(A|H) \wedge (\overline{H} \vee A) = A|H$; then by Definition 7 we have

$$(A|H)|(\overline{H} \vee A) = (A|H) \wedge (\overline{H} \vee A) + \mu\overline{A}H = A|H + \mu\overline{A}H = \begin{cases} 1, & \text{if } AH \text{ is true.} \\ \mu, & \text{if } \overline{A}H \text{ is true,} \\ x, & \text{if } \overline{H} \text{ is true.} \end{cases} \tag{23}$$

By coherence, $\mu \in [x, 1]$ and hence the iterated conditional $(A|H)|(\overline{H}\vee A)$ does not coincide with the constant $A|AH = 1$, thus the Import-Export principle does not hold. Moreover, when $P(\overline{H} \vee A) > 0$ from (12) it follows that

$$\mathbb{P}[(A|H)|(\overline{H} \vee A)] = \frac{P(A|H)}{P(\overline{H} \vee A)}.$$

We observe that the iterated conditional $(A|H)|(\overline{H} \vee A)$ is associated with the inference from the disjunction $\overline{H} \vee A$ to the conditional event $A|H$ (see Section 4.4). A probabilistic analysis of constructive and non-constructive inferences from the disjunction $A \vee B$ to the conditional event $B|\bar{A}$ has been given in Gilio and Over 2012.

More in general, given two conditional events $A|H$ and $B|K$, with $A|H \subseteq B|K$, defining $P(A|H) = x$, $P(B|K) = y$ and $\mathbb{P}[(A|H)|(B|K)] = \mu$, it holds that

$$(A|H)|(B|K) = (A|H) \wedge (B|K) + \mu\bar{B}|K = A|H + \mu\bar{B}|K.$$

Then, by (12), $\mu y = x$ and when $y > 0$ it follows that $\mu = \frac{x}{y}$, i.e.,

$$\mathbb{P}[(A|H)|(B|K)] = \frac{P(A|H)}{P(B|K)}, \quad \text{(if } A|H \subseteq B|K \text{ and } P(B|K) > 0).$$

4.3 Some p-valid Inference Rules

In Gilio, Pfeifer and Sanfilippo 2020, Theorem 8, it is shown that the p-entailment of a conditional event $E_3|H_3$ from a p-consistent family $\{E_1|H_1, E_2|H_2\}$ is equivalent to the condition $(E_3|H_3)|((E_1|H_1) \wedge (E_2|H_2)) = 1$, i.e., to the condition that the set of possible values of $(E_3|H_3)|((E_1|H_1) \wedge (E_2|H_2))$ is the singleton $\{1\}$. In Table 3 we illustrate some p-valid inference rules and the associated iterated conditionals which are equal to 1.

Inference rule	$\{E_1\|H_1, E_2\|H_2\} \Rightarrow_p E_3\|H_3$	$(E_3\|H_3)\|((E_1\|H_1) \wedge (E_2\|H_2)) = 1$
And	$\{B\|A, C\|A\} \Rightarrow_p BC\|A$	$(BC\|A)\|(BC\|A) = 1$
Cut	$\{C\|AB, B\|A\} \Rightarrow_p C\|A$	$(C\|A)\|(BC\|A) = 1$
CM	$\{C\|A, B\|A\} \Rightarrow_p C\|AB$	$(C\|AB)\|(BC\|A) = 1$
Or	$\{C\|A, C\|B\} \Rightarrow_p C\|(A \vee B)$	$(C\|(A \vee B))\|((C\|A) \wedge (C\|B)) = 1$
Modus Ponens	$\{C\|A, A\} \Rightarrow_p C$	$C\|AC = 1$
Modus Tollens	$\{C\|A, \bar{C}\} \Rightarrow_p \bar{A}$	$\bar{A}\|((C\|A) \wedge \bar{C}) = 1$
Bayes	$\{E\|AH, H\|A\} \Rightarrow_p H\|EA$	$(H\|EA)\|(EH\|A) = 1$

Table 3: *Some p-valid inference rules and their associated iterated conditionals.*

4.4 Some Non-p-valid Inference Rules

In this section we consider some non-p-valid inference rules, by showing that the associated iterated conditionals are not equal to 1.

Contraposition. Contraposition is not p-valid, that is the premise $\{C|A\}$ does not p-entail the conclusion $\bar{A}|\bar{C}$. Thus, from (18), $(\bar{A}|\bar{C})|(C|A) \neq 1$. Indeed, by setting, $P(C|A) = x$, $P(\bar{A}|\bar{C}) = y$, $\mathbb{P}[(C|A) \wedge (\bar{A}|\bar{C})] = z$, $\mathbb{P}[(\bar{A}|\bar{C})|(C|A)] = \mu$, it holds that

$$(\bar{A}|\bar{C})|(C|A) = (\bar{A}|\bar{C}) \wedge (C|A) + \mu(1 - C|A) = \begin{cases} y, & \text{if } AC \text{ is true,} \\ \mu, & \text{if } A\bar{C} \text{ is true,} \\ z, & \text{if } \bar{A}C \text{ is true,} \\ x + \mu(1-x), & \text{if } \bar{A}\bar{C} \text{ is true,} \end{cases}$$

which does not coincide with 1. For instance, by recalling that $\max\{x + y - 1, 0\} \leq z \leq \min\{x, y\}$, when $x = 1$ it holds that $z = y$. Then the iterated conditional becomes

$$(\bar{A}|\bar{C})|(C|A) = \begin{cases} y, & \text{if } C \text{ is true,} \\ \mu, & \text{if } A\bar{C} \text{ is true,} \\ 1, & \text{if } \bar{A}\bar{C} \text{ is true,} \end{cases}$$

with μ being coherent, for every $\mu \in [y, 1]$.
Strengthening the Antecedent. Strengthening of the antecedent is not p-valid, that is the premise $\{C|A\}$ does not p-entail the conclusion $C|AB$. Thus, from (18), $(C|AB)|(C|A) \neq 1$. Indeed, by setting $P(C|A) = x$, $P(C|AB) = y$, $\mathbb{P}[(C|AB) \wedge (C|A)] = z$, $\mathbb{P}[(C|AB)|(C|A)] = \mu$, it holds that

$$(C|AB)|(C|A) = (C|AB) \wedge (C|A) + \mu(1 - C|A) =$$
$$= \begin{cases} 1, & \text{if } ABC \text{ is true,} \\ \mu, & \text{if } AB\bar{C} \text{ is true,} \\ y, & \text{if } A\bar{B}C \text{ is true,} \\ \mu, & \text{if } A\bar{B}\bar{C} \text{ is true,} \\ z + \mu(1-x), & \text{if } \bar{A}BC \text{ is true,} \\ z + \mu(1-x), & \text{if } \bar{A}B\bar{C} \text{ is true,} \\ z + \mu(1-x), & \text{if } \bar{A}\bar{B} \text{ is true.} \end{cases}$$

By linearity of prevision it holds that

$$\mu = \mathbb{P}[(C|AB)|(C|A)] = \mathbb{P}[(C|AB) \wedge (C|A) + \mu(1 - C|A)] = z + \mu(1-x).$$

Then,

$$(C|AB)|(C|A) = \begin{cases} 1, & \text{if } ABC \text{ is true,} \\ y, & \text{if } A\bar{B}C \text{ is true,} \\ \mu, & \text{if } \bar{A} \vee \bar{C} \text{ is true,} \end{cases}$$

with μ being coherent, for every $\mu \in [y, 1]$.
From Disjunction to Conditional. Based on (18), the inference of a conditional $C|A$ from the associated material conditional $\bar{A} \vee C$ is not p-valid because, as shown in (23), the iterated conditional $(C|A)|(\bar{A} \vee C)$ does not coincide with 1.
Transitivity. Transitivity is not p-valid, that is the set of conditionals $\{C|B, B|A\}$ does not p-entail the conclusion $C|A$. Thus, from (20), $(C|A)|((C|B) \wedge (B|A))$ does not coincide with 1. Indeed, defining $P(B|A) = x$, $P(BC|A) = y$, $\mathbb{P}[(C|B) \wedge (B|A) \wedge (C|A)] = w$, $\mathbb{P}[(C|B) \wedge (B|A)] = z$, we have

$$(C|B) \wedge (B|A) \wedge (C|A) = (C|B) \wedge (BC|A) = \begin{cases} 1, & \text{if } ABC \text{ is true,} \\ 0, & \text{if } AB\bar{C} \text{ is true,} \\ 0, & \text{if } A\bar{B}C \text{ is true,} \\ 0, & \text{if } A\bar{B}\bar{C} \text{ is true,} \\ y, & \text{if } \bar{A}BC \text{ is true,} \\ 0, & \text{if } \bar{A}B\bar{C} \text{ is true,} \\ w, & \text{if } \bar{A}\bar{B} \text{ is true,} \end{cases} \tag{24}$$

and

$$(C|B) \wedge (B|A) = \begin{cases} 1, & \text{if } ABC \text{ is true,} \\ 0, & \text{if } AB\bar{C} \text{ is true,} \\ 0, & \text{if } A\bar{B}C \text{ is true,} \\ 0, & \text{if } A\bar{B}\bar{C} \text{ is true,} \\ x, & \text{if } \bar{A}BC \text{ is true,} \\ 0, & \text{if } \bar{A}B\bar{C} \text{ is true,} \\ z, & \text{if } \bar{A}\bar{B} \text{ is true.} \end{cases} \tag{25}$$

Then, by setting $\mu = \mathbb{P}[(C|A)|((C|B) \wedge (B|A))]$, it holds that

$$(C|A)|((C|B) \wedge (B|A)) = (C|B) \wedge (B|A) \wedge (C|A) + \mu(1 - (C|B) \wedge (B|A)) =$$
$$= \begin{cases} 1, & \text{if } ABC \text{ is true,} \\ \mu, & \text{if } AB\overline{C} \text{ is true,} \\ \mu, & \text{if } A\overline{B}C \text{ is true,} \\ \mu, & \text{if } A\overline{B}\overline{C} \text{ is true,} \\ y + \mu(1-x), & \text{if } \overline{A}BC \text{ is true,} \\ \mu, & \text{if } \overline{A}B\overline{C} \text{ is true,} \\ w + \mu(1-z), & \text{if } \overline{A}\overline{B} \text{ is true.} \end{cases} = \begin{cases} 1, & \text{if } ABC \text{ is true,} \\ y + \mu(1-x), & \text{if } \overline{A}BC \text{ is true,} \\ \mu, & \text{if } \overline{B} \vee \overline{C} \text{ is true} \end{cases}$$

because, by linearity of prevision, $\mu = w + \mu(1 - z)$. As we can see the iterated conditional $(C|A)|((C|B) \wedge (B|A))$ does not coincide with 1. Indeed, when $\overline{A}BC$ is true, the iterated conditional assumes the value $y + \mu(1 - x)$ which is equal to 0 when $(x, y) = (1, 0)$; then

$$(C|A)|((C|B) \wedge (B|A)) = \begin{cases} 1, & \text{if } ABC \text{ is true,} \\ 0, & \text{if } \overline{A}BC \text{ is true,} \\ \mu, & \text{if } \overline{B} \vee \overline{C} \text{ is true,} \end{cases}$$

with μ being coherent, for every $\mu \in [0, 1]$.

On Combining Evidence (Boole). We illustrate an example introduced in Boole 1857. In Hailperin 1996 it is shown that given a coherent assessment (x, y) on $\{C|A, C|B\}$, the extension $P(C|AB) = \xi$ is coherent for every $\xi \in [0, 1]$. Thus, the inference of $C|AB$ from the p-consistent family $\{C|A, C|B\}$ is not p-valid (see also Gilio and Sanfilippo 2019). We verify below that the associated iterated conditional $(C|AB)|((C|A) \wedge (C|B))$ does not coincide with 1. We set $P(C|A) = x$, $P(C|B) = y$, $\mathbb{P}((C|A) \wedge (C|B)) = z$. We obtain

$$(C|A) \wedge (C|B) = \begin{cases} 1, & \text{if } ABC \text{ is true,} \\ 0, & \text{if } (A \vee B)\overline{C} \text{ is true,} \\ x, & \text{if } \overline{A}BC \text{ is true,} \\ y, & \text{if } A\overline{B}C \text{ is true,} \\ z, & \text{if } \overline{A}\overline{B} \text{ is true.} \end{cases} \tag{26}$$

Moreover, by defining $\mathbb{P}[(C|A) \wedge (C|AB)] = u$, $\mathbb{P}[(C|B) \wedge (C|AB)] = v$ and $\mathbb{P}[(C|A) \wedge (C|B) \wedge (C|AB)] = t$, we obtain

$$(C|A) \wedge (C|B) \wedge (C|AB) = \begin{cases} 1, & \text{if } ABC \text{ is true,} \\ 0, & \text{if } (A \vee B)\overline{C} \text{ is true,} \\ u, & \text{if } \overline{A}BC \text{ is true,} \\ v, & \text{if } A\overline{B}C \text{ is true,} \\ t, & \text{if } \overline{A}\overline{B} \text{ is true.} \end{cases}$$

Then, by setting $\mu = \mathbb{P}[(C|AB)|((C|A) \wedge (C|B))]$, it holds that

$$(C|AB)|((C|A) \wedge (C|B)) = (C|A) \wedge (C|B) \wedge (C|AB) + \mu(1-(C|A) \wedge (C|B)) =$$

$$= \begin{cases} 1, & \text{if } ABC \text{ is true,} \\ \mu, & \text{if } (A \vee B)\overline{C} \text{ is true,} \\ u+\mu(1-x), & \text{if } \overline{A}BC \text{ is true,} \\ v+\mu(1-y), & \text{if } A\overline{B}C \text{ is true,} \\ t+\mu(1-z), & \text{if } \overline{A}\overline{B} \text{ is true.} \end{cases}$$

By linearity of prevision $\mu = t + \mu(1-z)$; then

$$(C|AB)|((C|A) \wedge (C|B)) = \begin{cases} 1, & \text{if } ABC \text{ is true,} \\ u+\mu(1-x), & \text{if } \overline{A}BC \text{ is true,} \\ v+\mu(1-y), & \text{if } A\overline{B}C \text{ is true,} \\ \mu, & \text{if } A\overline{C} \vee B\overline{C} \vee \overline{A}\overline{B} \text{ is true,} \end{cases}$$

which does not coincide with 1. We observe that, if we replace the conclusion $C|AB$ by $C|(A \vee B)$, we obtain the well-known p-valid Or rule.

We recall that the *Affirmation of the Consequent* and the *Denial of the Antecedent* are other non p-valid inference rules; thus, the associated iterated conditionals are not equal to 1 (Gilio, Pfeifer and Sanfilippo 2020). In Table 4 we list the previous non-p-valid inference rules and their associated iterated conditionals which do not coincide with 1. We denote by $\mathcal{C}(\mathcal{F})$ the conjunction of the conditional events in the set of premises $\mathcal{F}$.

Inference rule	$\mathcal{F} \not\Rightarrow_p E\|H$	$(E\|H)\|\mathcal{C}(\mathcal{F})) \neq 1$
Contraposition	$C\|A \not\Rightarrow_p \overline{A}\|\overline{C}$	$(\overline{A}\|\overline{C})\|(C\|A) \neq 1$
Strengthening the antecedent	$C\|A \not\Rightarrow_p C\|AB$	$(C\|AB)\|(C\|A) \neq 1$
From disjunction to conditional	$\overline{A} \vee C \not\Rightarrow_p C\|A$	$(C\|A)\|(\overline{A} \vee C) \neq 1$
Transitivity	$\{C\|B, B\|A\} \not\Rightarrow_p C\|A$	$(C\|A)\|((C\|B) \wedge (B\|A)) \neq 1$
Combining evidence	$\{C\|A, C\|B\} \not\Rightarrow_p C\|AB$	$(C\|AB)\|((C\|A) \wedge (C\|B)) \neq 1$
Affirmation of the Consequent	$\{C\|A, C\} \not\Rightarrow_p A$	$(A)\|((C\|A) \wedge C) \neq 1$
Denial of the antecedent	$\{C\|A, \overline{A}\} \not\Rightarrow_p \overline{C}$	$(\overline{C})\|((C\|A) \wedge \overline{A}) \neq 1$

Table 4: *Some non p-valid inference rules and their associated iterated conditionals.*

5. Conclusions

In this paper we have illustrated the notions of conjoined, disjoined and iterated conditionals introduced in recent papers, in the setting of coherence. These objects are defined as suitable conditional random quantities, with a finite set of possible values in the interval $[0,1]$. We have motivated our definitions by examining the experiment of flipping a coin twice. We have shown that the well known probabilistic properties valid for unconditional events continue to hold when replacing events by conditional events. We have examined several, intuitively acceptable, compound sentences on conditionals, by providing for them a formal interpretation in terms of iterated conditionals. We have discussed the Import-Export principle, which is not valid in our approach, by examining in particular the iterated conditional $(A|H)|(\overline{H} \vee A)$. Finally, we have illustrated, in terms of suitable iterated conditionals, several well known, p-valid and non p-valid, inference rules. With each inference rule, denoting by $E|H$ the conclusion and by $\mathcal{F}$ the set of premises, we have associated the iterated conditional $(E|H)|\mathcal{C}(\mathcal{F})$ where $\mathcal{C}(\mathcal{F})$ is the conjunction of the conditional events in $\mathcal{F}$. In Table 3 we recalled some well known p-valid inference rules, characterized by the property that $(E|H)|\mathcal{C}(\mathcal{F}) = 1$. Finally, we have examined some non p-valid inference rules, listed in Table 4, by verifying that for each of them the associated iterated conditional does not coincide with the constant 1.[1]

References

Adams, E.W. 1975, *The Logic of Conditionals*, Dordrecht: Reidel.

Baratgin, J., Politzer, G., Over, D. et al. 2018, "The Psychology of Uncertainty and Three-Valued Truth Tables", *Frontiers in Psychology*, 9, 1479.

Benferhat, S., Dubois, D. and Prade, H. 1997, "Nonmonotonic Reasoning, Conditional Objects and Possibility Theory", *Artificial Intelligence*, 92, 259-276.

Biazzo, V., Gilio, A., Lukasiewicz, T. et al. 2005, "Probabilistic Logic under Coherence: Complexity and Algorithms", *Annals of Mathematics and Artificial Intelligence*, 45, 1-2, 35-81.

Boole, G. 1857, "On the Application of the Theory of Probabilities to the Question of the Combination of Testimonies or Judgments", *Transactions of the Royal Society of Edinburgh*, 21, 4, 597-653.

Bradley, R. 2012, "Multidimensional Possible-World Semantics for Conditionals", *The Philosophical Review*, 121, 4, 539-71.

Calabrese, P. 1987, "An Algebraic Synthesis of the Foundations of Logic and Probability", *Information Sciences*, 42, 3, 187-237.

Calabrese, P. 2017, *Logic and Conditional Probability: A Synthesis*, London: College Publications.

Ciucci, D. and Dubois, D. 2012, "Relationships between Connectives in Three-Valued Logics", in Greco, S., Bouchon-Meunier, B., Coletti, G. et al. (eds.), *Advances in Computational Intelligence*, Proceedings of the 14th International Conference on Information Processing and Management of Uncertainty in Knowledge-Based Systems, IPMU2012, Catania, Italy, July 2012, CCIS 297, Berlin: Springer, 633-42.

Ciucci, D. and Dubois, D. 2013, "A Map of Dependencies among Three-Valued Logics", *Information Sciences*, 250, 162-77.

Coletti, G., Scozzafava, R. and Vantaggi, B. 2013, "Coherent Conditional Probability, Fuzzy Inclusion and Default Rules", in Yager, R.R., Abbasov, A.M., Reformat, M.Z. et al. (eds.), *Soft Computing: State of the Art Theory and Novel Applications*, Berlin: Springer, 193-208.

Coletti, G., Scozzafava, R. and Vantaggi, B. 2015, "Possibilistic and Probabilistic Logic under Coherence: Default Reasoning and System P", *Mathematica Slovaca*, 65, 4, 863-90.

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Cruz, N. 2020, “Deduction from Uncertain Premises?”, in Elqayam, Douven, Evans et al. 2020, 27-41.

de Finetti, B. 1936, “La logique de la probabilité”, in, *Actes du Congrès International de Philosophie Scientifique*, Vol. IV, Paris: Hermann et C.ie, 1-8.

de Finetti, B. 1980, “Foresight: Its Logical laws, Its Subjective Sources”, in Kyburg Jr., H.E. and Smokler, H.E. (eds.), *Studies in Subjective Probability*, 2nd edition, Huntington, NY: Krieger, 93-158.

de Finetti, B. 1995, “The Logic of Probability”, *Philosophical Studies*, 77, 181-90.

Douven, I. and Dietz, R. 2011, “A Puzzle about Stalnaker’s Hypothesis”, *Topoi*, 30, 31-7.

Douven, I., Elqayam, S., Singmann, H. et al. 2019, “Conditionals and Inferential Connections: Toward a New Semantics”, *Thinking* & *Reasoning*, 26, 1-41.

ECSQARU 2021, *Symbolic and Quantitative Approaches to Reasoning with Uncertainty: Proceedings of the 16th European Conference*, Lecture Notes in Computer Science, Berlin-Heidelberg: Springer, forthcoming.

Edgington, D. 1995, “On Conditionals”, *Mind*, 104, 235-329.

Edgington, D. 2020, “Compounds of Conditionals, Uncertainty, and Indeterminacy”, in Elqayam, Douven, Evans et al. 2020, 42-56.

Elqayam, S., Douven, I., Evans, J.S.B.T. et al. (eds.) 2020, *Logic and Uncertainty in the Human Mind: A Tribute to David E.Over*, London: Routledge.

Flaminio, T., Godo, L. and Hosni, H. 2020, “Boolean Algebras of Conditionals, Probability and Logic”, *Artificial Intelligence*, 286, URL: https://www.sciencedirect.com/science/article/pii/S000437022030103X.

Gilio, A. 2002, “Probabilistic Reasoning under Coherence in System P”, *Annals of Mathematics and Artificial Intelligence*, 34, 5-34.

Gilio, A. and Over, D. 2012, “The Psychology of Inferring Conditionals from Disjunctions: A Probabilistic Study”, *Journal of Mathematical Psychology*, 56, 118-31.

Gilio, A., Over, D., Pfeifer, N. et al. 2017, “Centering and Compound Conditionals under Coherence”, in Ferraro, M.B., Giordani, P., Vantaggi, B. et al. (eds.), *Soft Methods for Data Science*, Collection of selected peer reviewed papers presented at the 8th International Conference on Soft Methods in Probability and Statistics (SMPS 2016) held in Rome (Italy), Advances in Intelligent Systems and Computing 456, Berlin: Springer, 253-60.

Gilio, A., Pfeifer, N. and Sanfilippo, G. 2020, “Probabilistic Entailment and Iterated Conditionals”, in Elqayam, Douven, Evans et al. 2020, 71-101.

Gilio, A. and Sanfilippo, G. 2013a, “Conjunction, Disjunction and Iterated Conditioning of Conditional Events”, in Kruse, R., Berthold, M.R., Moewes, C. et al. (eds.), *Synergies of Soft Computing and Statistics for Intelligent Data Analysis*, Vol. 190, Berlin: Springer, 399-407.

Gilio, A. and Sanfilippo, G. 2013b, “Conditional Random Quantities and Iterated Conditioning in the Setting of Coherence”, in van der Gaag, L.C. (ed.), *Symbolic and Quantitative Approaches to Reasoning with Uncertainty*, Proceedings of the 12th European Conference, ECSQARU 2013, Lecture Notes in Computer Science 7958, Berlin-Heidelberg: Springer, 218-29.

Gilio, A. and Sanfilippo, G. 2013c, "Probabilistic Entailment in the Setting of Coherence: The Role of Quasi Conjunction and Inclusion Relation", *International Journal of Approximate Reasoning*, 54, 4, 513-25.

Gilio, A. and Sanfilippo, G. 2013d, "Quasi Conjunction, Quasi Disjunction, T-norms and T-conorms: Probabilistic Aspects", *Information Sciences*, 245, 146-67.

Gilio, A. and Sanfilippo, G. 2014, "Conditional Random Quantities and Compounds of Conditionals", *Studia Logica*, 102, 4, 709-29.

Gilio, A. and Sanfilippo, G. 2017, "Conjunction and Disjunction Among Conditional Events", in Benferhat, S., Tabia, K. and Ali, M. (eds.), *Advances in Artificial Intelligence: From Theory to Practice*, Proceedings of the 30th International Conference on Industrial Engineering and Other Applications of Applied Intelligent Systems, IEA/AIE 2017, Arras, France, June 27-30, 2017, Part II, Lecture Notes in Computer Science 10351, Springer, 85-96, URL: https://doi.org/10.1007/978-3-319-60045-1%5C_11.

Gilio, A. and Sanfilippo, G. 2019, "Generalized Logical Operations among Conditional Events", *Applied Intelligence*, 49, 1, 79-102.

Gilio, A. and Sanfilippo, G. 2020, "Algebraic Aspects and Coherence Conditions for Conjoined and Disjoined Conditionals", *International Journal of Approximate Reasoning*, 126, 98-123.

Gilio, A. and Sanfilippo, G. 2021a, "Compound Conditionals, Fréchet-Hoeffding Bounds, and Frank t-Norms", *International Journal of Approximate Reasoning*, 36, 168-200.

Gilio, A. and Sanfilippo, G. 2021b, "Iterated Conditionals and Characterization of P-Entailment", in ECSQARU 2021.

Goodman, I.R. and Nguyen, H.T. 1988, "Conditional Objects and the Modeling of Uncertainties", in Gupta, M.M. and Yamakawa, T. (eds.), *Fuzzy Computing: Theory, Hardware, and Applications*, Amsterdam: North-Holland Publishing Company, 119-38.

Goodman, I.R., Nguyen, H.T. and Walker, E.A. 1991, *Conditional Inference and Logic for Intelligent Systems: A Theory of Measure-Free Conditioning*, Amsterdam: North-Holland Publishing Company.

Hailperin, T. 1996, *Sentential Probability Logic: Origins, Development, Current Status, and Technical Applications*, Bethlehem: Lehigh University Press.

Kaufmann, S. 2009, "Conditionals Right and Left: Probabilities for the Whole Family", *Journal of Philosophical Logic*, 38, 1-53.

Knauff, M. and Spohn, W. (eds.) 2021, *Handbook of Rationality*, Cambridge, MA: The MIT Press, forthcoming.

Lewis, D. 1976, "Probabilities of Conditionals and Conditional Probabilities", *The Philosophical Review*, 85, 3, 297-315.

McGee, V. 1989, "Conditional Probabilities and Compounds of Conditionals", *Philosophical Review*, 98, 485-541.

Milne, P. 1997, "Bruno de Finetti and the Logic of Conditional Events", *British Journal for the Philosophy of Science*, 48, 195-232.

Mura, A. 2011, "Towards a New Logic of Indicative Conditionals", *Logic and Philosophy of Science*, 9, 17-31.

Nguyen, H.T. and Walker, E.A. 1994, "A History and Introduction to the Algebra of Conditional Events and Probability Logic", *IEEE Transactions on Systems, Man, and Cybernetics*, 24, 12, 1671-5.

Over, D. and Cruz, N. 2021, "Suppositional Theory of Conditionals and Rationality", in Knauff and Spohn 2021.

Pelessoni, R. and Vicig, P. 2014, "The Goodman-Nguyen Relation within Imprecise Probability Theory", *International Journal of Approximate Reasoning*, 55, 8, 1694-707.

Pfeifer, N. 2021, "Probability Logic", in Knauff and Spohn 2021.

Pfeifer, N. and Sanfilippo, G. 2021, "Interpreting Connexive Principles in Coherence-Based Probability Logic", in ECSQARU 2021.

Ramsey, F. 1990, "General Propositions and Causality", in Mellor, D.H. (ed.), *Philosophical Papers*, Cambridge: Cambridge University Press, 145-63.

Sanfilippo, G. 2018, "Lower and Upper Probability Bounds for Some Conjunctions of Two Conditional Events", in Ciucci, D., Pasi, G. and Vantaggi, B. (eds.), *Scalable Uncertainty Management*, Proceedings of the 12th International Conference, SUM 2018, Milan, Italy, October 3-5, 2018, Lecture Notes in Computer Science 11142, Heidelberg: Springer, 260-75.

Sanfilippo, G., Gilio, A., Over, D. et al. 2020, "Probabilities of Conditionals and Previsions of Iterated Conditionals", *International Journal of Approximate Reasoning*, 121, 150-73.

Sanfilippo, G., Pfeifer, N. and Gilio, A. 2017, "Generalized Probabilistic Modus Ponens", in A., A., L., C. and Papini, O. (eds.), *Symbolic and Quantitative Approaches to Reasoning with Uncertainty*, Proceedings of the 14th European Conference, ECSQARU 2017, Lugano, Switzerland, July 10-14, 2017, Lecture Notes in Computer Science 10369, Springer, 480-90, URL: https://doi.org/10.1007/978-3-319-61581-3%5C_43.

Sanfilippo, G., Pfeifer, N., Over, D. et al. 2018, "Probabilistic Inferences from Conjoined to Iterated Conditionals", *International Journal of Approximate Reasoning*, 93, Supplement C, 103-118.

Discovering Early de Finetti's Writings on Trivalent Theory of Conditionals

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Abstract

The trivalent and functional theory of the truth of conditionals developed by Bruno de Finetti has recently gathered renewed interests, particularly from philosophical logic, psychology and linguistics. It is generally accepted that de Finetti introduced his theory in 1935. However, a reading of his first publications indicates an earlier conception of almost all his theory. We bring to light a manuscript and unknown writings, dating back to 1928 and 1932, detailing de Finetti's theory. The two concepts of thesis and hypothesis are presented as a cornerstone on which logical connectives are established in a 2-to-3 valued logic. The proposed generalisation of the bivalent material implication to the trivalent framework, based on the bivalent entailment is however different from the one that will be introduced in 1935. In these early writings de Finetti presents original results that will later be independently rediscovered by other researchers. In particular, the 'suppositional logic' developed by Theodore Hailperin in 1996 presents numerous similarities. Conversely, we consider the notion of validity proposed by Hailperin in line with de Finetti's approach. Overall we attribute the primacy of the trivalent theory to de Finetti; this early conception enabled him to take an original position and argue with Hans Reichenbach.

Keywords: De Finetti's pioneer contributions to conditionals, de Finetti's conditional, Trivalent semantics, 2-to-3 valued logic validity.

1. Introduction: De Finetti Formed Its Trivalent Theory of Conditionals before 1935

For 20 years the trivalent theory of the truth of conditionals, proposed by Bruno de Finetti (1906-1985), has gathered numerous interests in various fields such as Philosophical Logic (e.g. Milne 1997; Mura 2009; Vidal 2014; Égré, Rossi and Sprenger 2020a, 2020b), Linguistics (e.g. Rothschild 2014; Douven 2016; Lassiter 2020; Lassiter and Baratgin 2021), Artificial Intelligence (e.g. Dubois and Prade 1994; Coletti and Scozzafava 2002), Psychology (e.g. Baratgin, Over and Politzer

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2013, 2014; Baratgin and Politzer 2016; Baratgin, Politzer, Over et al. 2018; Nakamura, Shao, Baratgin et al. 2018; Politzer, Jamet and Baratgin 2020) and Didactics (e.g. Delli Rocili and Maturo 2013).

According to de Finetti (1936), the indicative conditional *If* E_2*,* E_1 is a tri-event which is true if E_2 and E_1 are true, is false if E_2 is true and E_1 is false, and is otherwise undefined ('null' or 'void' truth value). More importantly, the indicative conditional is always undefined when its antecedent is false. For de Finetti, a tri-event can be understood through an analogy with a conditional *bet on If* E_2*,* E_1. This bet is won when E_2, E_1 are realized, lost when E_2 is realized but not E_1 and called off when E_2 is not realized. De Finetti proposes a trivalent logic system superimposed on traditional bi-valued logic with in addition to conditional, the usual connectives of negation, conjunction, disjunction and (material) implication.

The notion of 'tri-event' is the first essential step in de Finetti's approach, to define conditional probability as the result of a coherent subjective assessment. In his famous lecture to *Institut Henri Poincaré* in Paris, de Finetti (1937: 13-14) first demonstrates that the subjective coherent evaluation of probabilities by a given individual of an event always ranges between 0 and 1 and that the sum of the assessed probabilities of incompatible events makes 1. De Finetti uses a demonstrative method that entails: (i) to define an analogical unconditional bet on events E_i; (ii) to write the linear equation system of gains as a function of stakes S_i and outlaid pays $-P_iS_i$ with P_i probabilities of E_i evaluated by a given individual; and (iii) to apply the coherence constraint (not to lose the bet for sure) on this linear equation system. He then generalises this method to define the conditional probability with the following four successive steps:

i. The tri-event *if* E_2 *then* E_1 is presented through the analogy with a conditional bet in the particular situation where E_1 implies E_2. The bet is won when E_1 (and therefore E_2), lost when E_2 and not E_1 and called off when not E_2.

ii. The expressions of three possible gains (G, G_1, G_2) for the three possible outcomes in function of their stakes (S, S_1 and S_2) and the outlaid pays (pS, p_1S_1, p_2S_2) give a linear equation system of three equations:

$$\text{(CP)} \begin{cases} G \ = (1-p)S + (1-p_1)S_1 + (1-p_2)S_2 \\ G_1 = -pS - p_1S_1 + (1-p_2)S_2 \\ G_2 = 0 - p_1S_1 - p_2S_2 \end{cases}$$

iii. The notion of coherence (not to lose the bet for sure) gives a constraint on the linear equation system CP (its determinant must be null otherwise the stakes can be set so that the gains have arbitrary values, possibly all positive) requiring the relation $p_1 = pp_2$.

iv. Considering the general case $E_1 {\cap} E_2$ (and not simply E_1 with E_1 implies E_2) de Finetti obtains the conditional probability $P\left(\frac{E_1}{E_2}\right) = \frac{P(E_2 {\cap} E_1)}{P(E_2)}$ from which the Bayes' rule is derived.[1]

It is commonly accepted that de Finetti introduced his tri-event theory in 1935 (de Finetti 1936), at the Sessions on *Induction and Probability* of the *First Congress for the Unity of Science* (International Congress of Scientific Philosophy) in Paris (see Galavotti 2018: for an in-depth analysis of these sessions). Later writings, make reference in a rather scattered way to his trivalent logic. De Finetti refers to it through an analogy with the bet schema to define the conditional event to illustrate the logic of uncertainty underlying probabilities (de Finetti 1980: 1164-1165) or again to discuss the quantum logic (see appendix of de Finetti [1970] 1975: 304-313).

However as underlined by Mura (2009), the idea of a third truth value as 'null' when the antecedent is false was already present in 1934 in the book *L'invenzione della verità* published posthumously in 2006 (see de Finetti [1934] 2006: 103). Hence, one may wonder when did de Finetti really conceive his theory? Evidence indicates it happened before 1931.

Indeed, from 1930s onward, numerous arguments in favour of a subjective logic more general than the traditional objective logic can be found in several early publications of de Finetti (see de Finetti 1930; de Finetti 1931; de Finetti 1933; de Finetti [1934] 2006). The elements described then would support the logic of probabilities, the exposure of the betting scheme in an unconditional framework and also the presentation of the notion of coherence. Notably in the Memoria, dated 'Rome june 4 1930', *Sul significato soggettivo della probabilità*, de Finetti (1931) already exposes the demonstrative method in an unconditional situation that will be generalized to the conditional bet in de Finetti (1937). Yet, de Finetti (1931) foresees in his conclusion that the same process may allow the definition of conditional probability:

> It will then be observed that no mention has ever been made here of subordinate probabilities (probability that an event occurs when another event is supposed to occur),

[1] For the sake of consistency with the rest of the document we use from now on the same notations and terminology used in the original de Finetti's manuscripts presented and discussed in section 2. Thus, we use: "$-$" for "negation" or "opposite", "$\cap$" and "$\cup$" for respectively "product" and "sum" (As noted by Mura (2009: 203), de Finetti 1936 uses neither the logic terms of "conjunction" nor "disjunction". De Finetti's goal is to construct a ternary logic that supports the subjective probability theory. Consequently he uses the same vocabulary as used in probability theory), ":=:" for equivalence and "=" defined as $a = b$:=: $a^{\cup} b \supset a_{\cap} b$. Tri-event are noted "$\frac{E_1}{E_2}$" instead of the modern notation "$E_1 \mid E_2$". We remain faithful as much as possible to the term "subordinate" traditionally used in Italian and French mathematical papers at that time rather than to use the modern term "conditional". These two terms are considered similar in the literature (e.g. de Finetti 1967). A subordinate conditional clause is used when a fact or action is necessary before another fact or action is carried out. The subordinate clause is more general. It conveys two kinds of information: foreground information (i.e. open information), the communication of which is the subordinate's actual task, and background information (i.e. implied, epiphenomenal, incidental information concerning the subject's opinion of the speaker) generally known as a "presupposition" (Ducrot 1969). De Finetti's tri-event can account for the "elementary presupposition" (e.g. Beaver and Krahmer 2001; Ducrot [1980] 2008). De Finetti seemed to agree with this idea (see Mura's notes 9 and 12, 174-175 in de Finetti [1979] 2008).

> of the relative theorem of compound probabilities, and of the resulting concept of independent events. These notions are much more delicate and cannot be ordinarily judged, and that it is not at all necessary to introduce them to start with. One can, and indeed it is advisable, if one wants to make the concepts clear, first develop the theory of the probabilities of an event, a theory of which we have given all the foundations here, and then leave to a second time the extension of the calculation of probabilities to subordinate events, an extension that needs support, definitions and explanations that are completely new and conceptually interesting. This topic will also be the subject of other work. We observe, however, from now on, that, if we want to be satisfied with a definition without psychological content, as usually given, we would already have all the elements to define 'formally' the subordinate probability, calling 'probability of E_1 subordinate to E_2' = $\frac{P(E_1 \cap E_2)}{P(E_2)}$. From it, would immediately result, the theorem of the compound probabilities:
>
> $$P(E_1 \cap E_2) = P(E_2) \times P\left(\frac{E_1}{E_2}\right),$$
>
> if we indicate $P\left(\frac{E_1}{E_2}\right)$ the probability of E_1 subordinated to E_2; such theorem, however, would only be a concealed definition of the symbol $P\left(\frac{E_1}{E_2}\right)$. In the way of proceeding that we will develop and have announced here, we will instead give a direct psychological definition of subordinate probabilities thanks to which the theorem on compound probabilities (and therefore the 'formal' definition indicated here) results as a necessary consequence of the usual definition of coherence. And this is the only way of proceeding in accordance with our point of view (de Finetti 1931: 328-329, our translation).[2]

Therefore de Finetti has certainly developed, as early as 1930, the concept of tri-event confirming once again the assertion of Morini[3]

> that de Finetti's theory takes an almost definitive form since the very beginning of his research. It was between 1929 and 1931 that his theory of probability took shape, which he continued to defend throughout his rich scientific career. The almost 300 articles he wrote, the first of which were mostly mathematical in content, are a re-elaboration and deepening of the ideas he had conceived at the age of twenty (Morini 2007: 3-4, our translation).

De Finetti's original writings were acquired by the University of Pittsburgh and stored in the Archives of Scientific Philosophy, alongside other writings representing the so-called 'philosophy of science' of the last century (Ramsey, Carnap, Reichenbach, Hempel, Feigl and Salmon). Among these documents, two folders containing original writings on tri-events are of interest to us.

- *Box 6, Folder 2* entitled *"Logica plurivalente", 1927-1935* (see figure 1) and cited here as de Finetti 1927-1935. In addition to handwritten and typed drafts of de Finetti's (1936) presentation, it contains a correspondence with Hans Reichenbach (1891-1953) dated 1935 as well as original writings, text

[2] De Finetti (1931) uses E and E' instead of E_1 and E_2 and '.' instead of '$\cap$'.

[3] For example, de Finetti's concept of *random exchangeable sequences* dates back to 1930 (Bassetti and Regazzini 2008), as well as his criticism on countable additivity (Regazzini 2013).

and mixed drafts dated 1927.[4] Notably, there are 7 pages of typewritten text dated "Rome, Sunday September 16 1928" and titled: *l'EVENTO SUBORDINATO* [5] *come ente logico* [The subordinate event as a logical entity] de Finetti 1927-1935: 154-60, #'BD-06-02-55' and cited here as de Finetti 1928a, one manuscript page dated 'Rome, March 18 1928' titled *Logica degli eventi* [Logic of events] (de Finetti 1927-1935: 173, #'BD-06-02-66') and cited here as de Finetti 1928c and several draft sheets which were used to establish the demonstrations of the writings.[6]

- *Box 5, Folder 10* entitled *Lezioni sulla probabilità* [Lessons on probability], dated 1932-1933 and cited here as de Finetti 1932a. This folder corresponds to the manuscript of lectures that de Finetti gave in 1932-1933 at Trieste University.[7] The notion of tri-event is synthetically literally presented (26-28).

We will analyse how de Finetti's early approach and methods differ from the presentation of de Finetti 1936. We will underline the original results that were later rediscovered independently by other authors such as Theodore Hailperin (1915-2014) with his 'suppositional logic'. Conversely, we will propose Hailperin's notion of validity (Hailperin 1996, 2011) as compatible with de Finetti approach. Finally, the differences with Reichenbach's approach will be discussed.

2. The Subordinate Event as a Logical Entity

De Finetti (1936) starts with a long critical discourse on usual three-value logic and argues in favour of a generalization of the binary formal logic of 'ordinary' events to conditional events. It presents the trivalent logic system with truth tables for the different connectors as a 'perfect analogy' to two-valued logic. De Finetti then introduces the two operations called 'thesis' and 'hypothesis' in order to return to ordinary binary logic. In de Finetti 1928a, the presentation is inverted. After a short presentation of 'subordinated event', the notions of 'thesis' and 'hypothesis' are introduced. Thus across Part 2 to Part 7, de Finetti 1928a remains in a bivalent framework. De Finetti presents the third value only in Part 7 referring to these two unary operations. Part 10 concerns the link with probability theory and Parts 11-15 constitute an arithmetic analogy of de Finetti's trivalent logic. Most of the demonstrations can be found either in the manuscript or in the various drafts that preceded it.

[4] In 1927, Bruno de Finetti, a 20 years old student in Milan, graduated in applied mathematics. Shortly after he accepted a position in Rome at the *Instituto Centrale di Statistica*, chaired at that time by the famous Italian statistician Corrado Gini Cifarelli and Regazzini 1996; de Finetti, F. and Nicotra 2008. Among these documents, there are (i) two manuscript versions (one of which is dated "Milan, March 23 1927") of de Finetti 1927, (ii) three manuscript versions (one of which is dated "Milan, May 8 1927") of de Finetti 1928d, (iii) three manuscript versions (one of which is dated, "Rome April 23 1929") which seems to be a draft of de Finetti 1932b.

[5] Capitalized by the author.

[6] These drafts will be cited here as de Finetti 1928b with their page number and identifier (#) indicated at the top of the page.

[7] In 1931, de Finetti accepted an actuary position with *Assicurazioni Generali* in Trieste (see de Finetti, F. and Nicotra 2008).

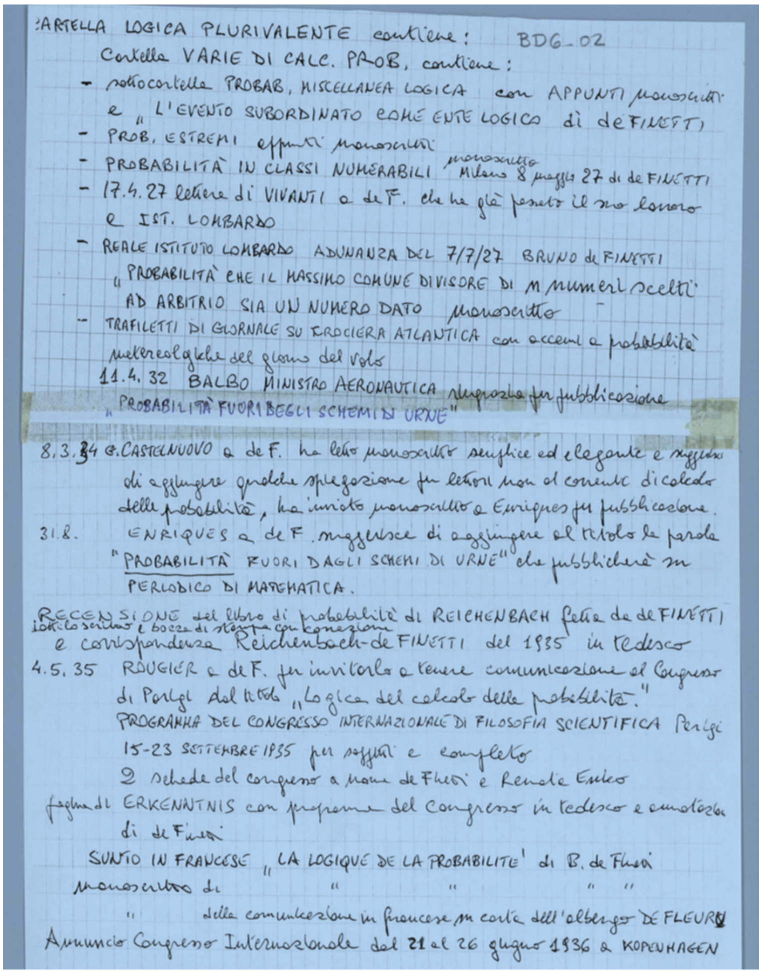

CARTELLA LOGICA PLURIVALENTE contiene: BDG_02
Cartella VARIE DI CALC. PROB. contiene:
- sottocartella PROBAB. MISCELLANEA LOGICA con APPUNTI manoscritti e „L'EVENTO SUBORDINATO COME ENTE LOGICO" di de FINETTI
- PROB. ESTREMI appunti manoscritti
- PROBABILITÀ IN CLASSI NUMERABILI manoscritto Milano 8 maggio 27 di de FINETTI
- 17.4.27 lettera di VIVANTI a de F. che ha già passato il suo lavoro a IST. LOMBARDO
- REALE ISTITUTO LOMBARDO ADUNANZA DEL 7/7/27 BRUNO de FINETTI „PROBABILITÀ CHE IL MASSIMO COMUNE DIVISORE DI m numeri scelti AD ARBITRIO SIA UN NUMERO DATO manoscritto
- TRAFILETTI DI GIORNALE SU CROCIERA ATLANTICA con accenni a probabilità meteorologiche del giorno del volo
11.4.32 BALBO MINISTRO AERONAUTICA ringrazia per pubblicazione „PROBABILITÀ FUORI DEGLI SCHEMI DI URNE"
8.3.34 G.CASTELNUOVO a de F. ha letto manoscritto semplice ed elegante e suggerisce di aggiungere qualche spiegazione per lettori non al corrente di calcolo delle probabilità, ha inviato manoscritto a Enriques per pubblicazione.
31.8. ENRIQUES a de F. suggerisce di aggiungere al titolo le parole "PROBABILITÀ FUORI DAGLI SCHEMI DI URNE" che pubblicherà in PERIODICO DI MATEMATICA.
RECENSIONE del libro di probabilità di REICHENBACH fatta da de FINETTI
articolo scritto e bozza di stampa con correzioni
e corrispondenza Reichenbach-de FINETTI del 1935 in tedesco
4.5.35 ROUGIER a de F. per invitarlo a tenere comunicazione al Congresso di Parigi dal titolo „Logica del calcolo delle probabilità."
PROGRAMMA DEL CONGRESSO INTERNAZIONALE DI FILOSOFIA SCIENTIFICA Parigi 15-23 SETTEMBRE 1935 per soggetti e completo
2 schede del congresso a nome de Finetti e Renata Emko
foglio di ERKENNTNIS con programma del Congresso in tedesco e annotazioni di de Finetti
SUNTO IN FRANCESE „LA LOGIQUE DE LA PROBABILITE' di B. de Finetti
manoscritto di " " " "
" della comunicazione in francese su carta dell'albergo DE FLEURUS
Annuncio Congresso Internazionale dal 21 al 26 giugno 1936 a KOPENHAGEN

Figure 1: Document cover of Folder 2 *"Logica Plurivalente", 1927-1935*.

2.1 The Subordinated Event as a Subordinated Bet

Part 1 corresponds to a short introduction in which de Finetti underlines the difference between the implication of propositional logic and the 'subordinate' relation used for subordinate probability. He gives the truth values for the two relations according to the 'thesis' and the 'hypothesis'. He illustrates this point by introducing the example of a bet on the outcome of a coin toss (thesis). Such a bet is subordinated to the fact that the coin has indeed been launched (hypothesis). To our knowledge, this is the first written record where de Finetti presents the conditional bet.

> The statement 'if E_2 is true then E_1 is true' of logic, in symbols: $E_2 \supset E_1$, is a true proposition if the thesis and hypothesis are true, or if the hypothesis is false, it is false only if the hypothesis is true and the thesis is false. When we speak instead of the probability of an event subordinate to another, the statement 'if E_2 is true

then E_1 is true' has a very different value, having to be considered true if the thesis and hypothesis are true, false if the thesis is false and the hypothesis is true, and insignificant (neither true nor false) if the hypothesis is false. In fact, if one was to bet, for example, "if I throw a coin, it will show head", and then not throw the coin, one could not claim to have won the bet, although one's statement, understood as a logical deduction, is true, having a false proposition by hypothesis.

Therefore we have to consider a new logical entity: the subordinate affirmation (or also subordinate event, which has by conception original and useful applications (de Finetti 1928a: 1, underlined by the author, our translation).

The subordinate event $\frac{E_1}{E_2}$ has three possible values following those of E_1 and E_2 which are 2-valued statements of the bivalent logic (a bet can only consider the realisation or non realisation of an event E). So here we have, from the outset, the idea of a three-valued logic provisionally *superimposed* on the traditional two-valued logic. The 'subordination operation' can then be extended to situations where E_1 and E_2 are 'insignificant' when they are themselves subordinated events (see section 2.6). The comparison of implication and subordinate event is given in term of 'thesis' and 'hypothesis' (subsequently noted by de Finetti 1928a T and H) which in the citation seems synonymous of consequent E_1 and antecedent E_2. The truth or falsity of the implication is given by the definition of the entailment (noted $\leq$) where $E_2 \leq E_1$ if $E2$ is false or if E_1 and E_2 are true (thus $\leq$ can often be considered as an order relation with false < true).

(1) $E_2 \supset E_1 :=: E_2 \leq E_1$

In his 1932 course at Trieste University, de Finetti also introduces the subordinate event as a conditional bet. The third value is called 'indeterminate' rather than 'insignificant'. He compares the semantic tables for implication and subordinate event in a way that we illustrate in Table 1.

Up to now we have always talked about the probability of an E event that could only be true or false; we must now consider the more general case of an event—we shall say, of a subordinate event)—that can be either true, or false, or indeterminate.

To make the concept intuitive, let us return to betting: until now we have dealt with bets made in such a way that they were certainly either won or lost; it is frequent, however, under circumstances established as indeterminate that the bet is void. For example, in a bet on the outcome of a football match, it can be agreed that the bet is void in the event of draw. In such a case, what is the betting on? The following statement: "If one of the two teams wins, the victory will go to team A".

This statement differs from those considered so far in that there is a condition (that one of the two teams wins) which limits the field of possibility for which the bet is established; in it the statement 'the win is up to team A' is made subject to the condition premise, i.e. the hypothesis that 'one of the two teams wins'.

Be E_1 and E_2 two events, we will generally indicate with the symbol E_1/E_2 (or also, according to the typographical convenience of the single case, $\frac{E_1}{E_2}$; read: "E_1 is subordinate to E_2") the statement (subordinate event) that "if E_2 is supposed to be true then E_1 is therefore true"; so E_1/E_2 is indeterminate when E_2 is false (i.e. in the case $-E_2$), true when E_1 and E_2 are true (case $E_1{\cap}E_2$), false when E_2 is true and E_1 is false (case $E_2{\cap}{-E_1}$).

> We must not make the confusion, despite the analogy of its verbal formulation, between the subordinate event E_1/E_2 and the event or statement that is $E_2 \supset E_1$ in formal logic ("E_2 implies E_1"), and means $(-(E_{2\cap} - E_1))$, (it is therefore false if E_1 is false and E_2 is true, while it is true in any other case, i.e. as much as when E_2 and E_1 are both true or as when E_2 is false and E_1 is either true or false). The relation between the two concepts is however close: $E_2 \supset E_1$ means "E_1/E_2 is not false" (de Finetti 1932a: 26-27, underlined by the author, our translation with the notation of de Finetti 1928a).

There is a nuance with de Finetti (1928a) in the definition of the implication. The implication is considered as an operation on the subordinate event ("E_1/E_2 is not false")[8] to pass from a ternary situation to the traditional bivalent situation. De Finetti (1928) also gives here the traditional definition 2 of implication, which is equivalent to definition 1 in the bivalent framework (see however section 3.1):

(2) $E_2 \supset E_1 :=: E_1{}^{\cup} - E_2 :=: E_{1\cap}E_2{}^{\cup} - E_2$

Both connectives are presented below in Table 1. The 'insignificant' value is the consequence of the falsehood of the hypothesis. Unlike the 'true' and 'false' values it appears as an output and not as an input of the truth table.

E_2	E_1	$E_2 \supset E_1$	$\frac{E_1}{E_2}$
true	true	true	true
true	false	false	false
false	true	true	'insignificant' or 'indeterminate'
false	false	true	'insignificant' or 'indeterminate'

Table 1: Semantic tables for implication and subordination (de Finetti 1928a,1932).

The basic idea of de Finetti, to consider that indicative conditionals can have three values of truth, was rediscovered and developed by an important number of authors with interesting variations (for a review, see the supplementary material of Baratgin, Politzer, Over et al. 2018). For example Hailperin (1996: 35-36) introduces the same Table 1 with a third value called 'don't care'.

2.2 First Definition of Subordinate Event

In Part 2, de Finetti presents the notations and symbols used. A subordinate event $E\left(\frac{E_1}{E_2}\right)$ is called 'absolute event' when E_2 is true. In this case E corresponds to the 'not-subordinated' event E_1, which can be true or false.[9]

De Finetti takes the original symbols '$\oplus$' and '$\ominus$' respectively for 'true' and 'false'. This choice to associate truth with a '+' and falsehood with a '−' can easily be interpreted as an implicit reference to a gain of a bet schema. If I bet on an

[8] Recall that only the situations where E_1 and E_2 are true or false are considered.

[9] This term, also used in de Finetti 1932a, will be referred as "ordinary event" in de Finetti 1936 and de Finetti 1937.

event E that comes true, I win my bet and conversely if it does not come true, E is then wrong and I lose my bet.[10]

Thus, de Finetti 1928a gives the truth and falsehood definition of a subordinate event:[11]

$$(3) \quad (^*) \quad \frac{E_1}{E_2} = \oplus \;:=:\; E_{2\cap}E_1 = \oplus \text{ and } \frac{E_1}{E_2} = \ominus \;:=:\; E_{2\cap} - E_1 = \oplus$$

It is important to stress that de Finetti at that time remains within the bi-valued framework. No reference is made to a third value or to the situation where $-E_2 = \oplus$. De Finetti, does not give any justification for the definition 3. However de Finetti (1928b: draft #'BD6-02-61', 166), partitions E_1 and E_2 as the sum of their constituents:
$E_2 = E_{1\cap}E_2{}^{\cup} - E_{1\cap}E_2 = A^{\cup}B$ and $E_1 = E_{2\cap}E_1{}^{\cup} - E_{2\cap}E_1 = A^{\cup}C$
with $A = E_{1\cap}E_2$, $B = -E_{1\cap}E_2$ and $C = -E_{2\cap}E_1$. If we suppose $E_2 = \oplus$ then $E_1 = A = -B$ which gives a correct intuition for definition 3.

2.3 Thesis, Hypothesis, Irreducible Form and Subordinate Event Equivalent

Part 3 focuses on the definition of the 'hypothesis' and 'thesis' unary operators. The 'hypothesis' (H) is the "absolute event which is necessary and sufficient to occur for a subordinate event E to be true or false" and the 'thesis' (T) is the "absolute event that is necessary and sufficient to occur for a subordinate event E to be true".[12]

$$(4) \quad H(E) \;:=:\; (E = \oplus)^{\cup}(E = \ominus) \text{ and } T(E) \;:=:\; (E = \oplus)$$

De Finetti defines the subordinated event E which follows the hypothesis and thesis:

$$(5) \quad E = \frac{E_1}{E_2} \text{ we have } H(E) = E_2 \text{ and } T(E) = E_{1\cap}E_2$$

Thus

$$(6) \quad E = \frac{T(E)}{H(E)} = \frac{E_{1\cap}E_2}{E_2}$$

Form 6 corresponds to a simplified form 'analogous to fractions reduced to the minimum terms' that de Finetti (1932a) calls 'irreducible':

[10] In latter writings, de Finetti will modify these notations by taking the traditional conventions "T" and "F" (de Finetti 1936, 1975) or Boole's convention "1" and "0" (de Finetti 1967, 1975, 1980) for true and false respectively.

[11] The * sign put by the author to identify the relation is likely to underline the importance of this relation. We added a number for the sake of identification.

[12] In de Finetti 1936 the definitions for $H(E)$ will be "E is not null" (E does not have the third truth value). Here de Finetti remains in the bivalent framework because he has not yet defined the third value.

> Let us observe that $\frac{E_1}{E_2} = \frac{A_1}{A_2}$, that is $\frac{E_1}{E_2}$ and $\frac{A_1}{A_2}$ represent the same subordinate event, if and only if $E_2 = A_2$ and $E_{2\cap}E_1 = A_{2\cap}A_1$; in fact, depending on whether one is true, false, indeterminate, the other is equally true, false, indeterminate ($-E_2 = -A_2$, $E_{2\cap}E_1 = A_{2\cap}A_1$, $E_{2\cap} - E_1 = A_{2\cap} - A_1$). Therefore it is not necessary that $E_1 = A_1$; in particular, E_1 can always be substituted with $E_{2\cap}E_1$, thus reducing the expression of the event subordinate to the form that we will say <u>irreducible</u>. It is a matter of eliminating also in the statement the apparent inclusion of cases that go for excluded hypothesis : if, for example. If it had been said "if one of the two teams wins, team A does not lose" (E_2 = "one of the two teams wins", E_1="team A does not lose") we would have made no more no less the same statement (possibly a bet) than before, when it was said "if one of the two teams wins, team A wins" (being $E_{2\cap}E1$ = "one of the two teams wins" and "team A does not lose"="team A wins"): the difference is only formal, because in saying "team A does not lose" the case of a draw remains included in the sentence, which, however, in the whole of the subordinate statement, remains excluded by hypothesis.
>
> In a subordinate event, or subordinate statement, $\frac{E_1}{E_2}$, one can call <u>hypothesis</u> the event (or statement) E_2; <u>thesis</u> the event (or statement $E_{2\cap}E_1$; every subordinate event can be written in the form $\left(\frac{Thesis}{Hypothesis}\right)$, and this is the form we called <u>irreducible</u> (de Finetti 1932a: 27-28, underlined by the author, our translation with the notations of de Finetti 1928a).

Hailperin's suppositional normal form of E (Hailperin 1996, 2011) corresponds to de Finetti's irreducible form formulated with the constituents of E from its 'condensed' semantic table by considering only the values true and false for its atoms (as in Table 1) (Hailperin 1996; Hailperin 2011). $T(E)$ corresponds to the constituents of E that are true and $H(E)$ to those that are true or false. Each event E has a 'unique suppositional normal form'. Two subordinate events are 'equivalent' if their suppositional normal forms are the same (Hailperin 1996: 250).

Which is written with the notation of de Finetti (1928) as

$$(7)\quad \begin{array}{l} E :=: \dfrac{E_1}{E_2} :=: \dfrac{E_1'}{E_2'} :=: E' \text{ if and only if} \\ H(E) = E_2 :=: E_2' = H(E') \text{ and } T(E) = E_{2\cap}E_1 :=: E_{2\cap}'E_1' = T(E') \end{array}$$

with the 'Left Logical Equivalence' principle as corollary :

$$(8)\quad \text{If } H(E) = E_2 :=: E_2' = H(E') \text{ if and only if } \frac{E_1}{E_2} :=: \frac{E_1}{E_2'}$$

Part 4 is dedicated to the negation relations:

$$(9)\quad -E = \oplus :=: E = \ominus \text{ and } E = \ominus :=: -E = \oplus$$

De Finetti (1928a) deduces from the definition 4,[13] the hypothesis and thesis of the negation of E:[14]

[13]
$$\begin{aligned} H(-E) :=: (-E = \oplus)^{\cup}(-E = \ominus) \quad & :=: (E = \ominus)^{\cup}(E = \oplus) \\ T(-E) :=: (-E = \oplus) :=: (E = \ominus) \quad & :=: (E = \ominus)^{\cup}\left[(E = \oplus)_{\cap} - (E = \oplus)\right] \\ & :=: \left[(E = \oplus)^{\cup}(E = \ominus)\right]_{\cap}\left[(E = \ominus)^{\cup} - (E = \oplus)\right] \\ T(-E) \quad & :=: H(E)_{\cap} - T(E). \end{aligned}$$

[14] The "antithesis" $-T(E)$ in de Finetti (1970) 1974: 130 is also noted $\mathcal{L}$ in de Finetti (1928b: drafts #'BD6-02-66', 171 and #'BD6-02-68', 175).

(10) $H(-E) :=: H(E)$ and $T(-E) :=: H(E)_{\cap} - T(E)$

Thus

(11) $T(E)_{\cap}T(-E) = \ominus$ and $T(E)^{\cup}T(-E) = H(E)$

2.4 Sum and Product

Part 5 focuses on the sum and product of subordinate events (formulated here for only two subordinated events E_1 and E_2).

$$(12)\quad \begin{cases} (E_1{}^{\cup}E_2) = \oplus :=: (E_1 = \oplus)^{\cup}(E_2 = \oplus) & (E_{1\cap}E_2) = \oplus :=: (E_1 = \oplus)_{\cap}(E_2 = \oplus) \\ (E_1{}^{\cup}E_2) = \ominus :=: (E_1 = \ominus)_{\cap}(E_2 = \ominus) & (E_{1\cap}E_2) = \ominus :=: (E_1 = \ominus)^{\cup}(E_2 = \ominus) \end{cases}$$

$$(13)\quad \begin{cases} -(E_{1\cap}E_2) = (-E_1{}^{\cup} - E_2) \\ -(E_1{}^{\cup}E_2) = (-E_{1\cap} - E_2) \end{cases}$$

The product and sum for theses and hypotheses follow:

$$(14)\quad \begin{cases} T(E_{1\cap}E_2) = T(E_1)_{\cap}T(E_2) \\ T(E_1{}^{\cup}E_2) = T(E_1)^{\cup}T(E_2) \\ H(E_{1\cap}E_2) = [T(E_1)_{\cap}T(E_2)]^{\cup}T[-(E_{1\cap}E_2)] \\ \qquad\qquad = [T(E_1)_{\cap}T(E_2)]^{\cup}T[(-(E_1)^{\cup} - E_2)] \\ \qquad\qquad = [T(E_1)_{\cap}T(E_2)]^{\cup}T(-E_1)^{\cup}T(-E_2) \\ H(E_1{}^{\cup}E_2) = T(E_1{}^{\cup}E_2)^{\cup}[T(-E_1{}^{\cup}E_2)] \\ \qquad\qquad = T(E_1{}^{\cup}E_2)^{\cup}[T(-E_{1\cap} - E_2)] \\ \qquad\qquad = T(E_1)^{\cup}T(E_2)^{\cup}[T(-E_1)_{\cap}T(-E_2)] \end{cases}$$

De Finetti (1928a, 3) indicates 'These formulas, for $E = \frac{T(E)}{H(E)}$, give the complete expression of the sum and of the product'.[15] Thus with $E_1 = \frac{E_1'}{E_1''}$ and $E_2 = \frac{E_2'}{E_2''}$:

$$\begin{aligned} E_{1\cap}E_2 &= \frac{T(E_{1\cap}E_2)}{H(E_{1\cap}E_2)} \\ (15a)\qquad &= \frac{T(E_1)_{\cap}T(E_2)}{[T(E_1)_{\cap}T(E_2)]^{\cup}T(-E_1)^{\cup}T(-E_2)} \\ &= \frac{E_1'{}_{\cap}E_1''{}_{\cap}E_2'{}_{\cap}E_2''}{[E_1'{}_{\cap}E_1''{}_{\cap}E_2'{}_{\cap}E_2'']^{\cup}[-E_1'{}_{\cap}E_1''{}^{\cup} - E_{2\cap}{}'E_2'']} \end{aligned}$$

[15] The formulation 15b comes from de Finetti 1928b: draft #'BD6-02-67', 173. These formulas were rediscovered much later independently by Goodman, Nguyen and Walker (1991) and by Hailperin (1996).

$$
\begin{aligned}
&:=: \frac{E'_{1\cap}E''_{1\,\cap}E'_{2\cap}E''_2}{E''_{1\,\cap}E''^{\cup}_2 - E'_{1\cap}E''^{\cup}_1 - E_{2\cap}{}'E''_2}\\
&:=: \frac{E'_{1\cap}E'_2}{E''_{1\,\cap}E''^{\cup}_2 - E'_{1\cap}E''^{\cup}_1 - E_{2\cap}{}'E''_2}
\end{aligned}
$$

$$
\begin{aligned}
E_1{}^{\cup}E_2 &= \frac{T(E_1{}^{\cup}E_2)}{H(E_1{}^{\cup}E_2)}\\
&= \frac{T(E_1)^{\cup}T(E_2)}{T(E_1)^{\cup}T(E_2)^{\cup}\left[T(-E_1)_{\cap}T(-E_2)\right]}\\
\text{(15b)} \qquad &= \frac{E'_{1\cap}E''^{\cup}_1E'_{2\cap}E''_2}{\left[E'_{1\cap}E''^{\cup}_1E'_{2\cap}E''_2\right]^{\cup}\left[-E'_{1\cap}E''_{1\,\cap} - E'_{2\cap}E''_2\right]}\\
&:=: \frac{E'_{1\cap}E''^{\cup}_1E'_{2\cap}E''_2}{E'_{1\cap}E''^{\cup}_1E'_{2\cap}E''^{\cup}_2E''_{1\,\cap}E''_2}\\
&:=: \frac{E'^{\cup}_1E'_2}{E'_{1\cap}E''^{\cup}_1E'_{2\cap}E''^{\cup}_2E''_{1\,\cap}E''_2}
\end{aligned}
$$

De Finetti poses the 'well-known logical identities'[16]

$$
\text{(16)} \quad \begin{cases}(E_{1\cap}E_2)^{\cup}E_3 = (E_1{}^{\cup}E_3)_{\cap}(E_2{}^{\cup}E_3)\\ (E_1{}^{\cup}E_2)_{\cap}E_3 = (E_{1\cap}E_3)^{\cup}(E_{2\cap}E_3)\end{cases}
$$

In Part 6 de Finetti affirms that the hypotheses of the sum and the product of subordinate events are always contained (noted $:\supset:$) between the sum of the hypotheses and their product.[17]

$$
\text{(17)} \quad \begin{cases}H(E_1)_{\cap}H(E_2) \;:\supset:\; H(E_{1\cap}E_2) \;:\supset:\; H(E_1)^{\cup}H(E_2)\\ H(E_1)_{\cap}H(E_2) \;:\supset:\; H(E_1{}^{\cup}E_2) \;:\supset:\; H(E_1)^{\cup}H(E_2)\end{cases}
$$

with the corollary:

$$\text{if } H(E_1) = H(E_2) = E \text{ then } H(E_1)_{\cap}H(E_2) = H(E_1)^{\cup}H(E_2) = E.$$

[16] The demonstration can be read in de Finetti 1928b: draft #'BD6-02-65', 170:

$$
\begin{aligned}
T\left[(E_{1\cap}E_2)^{\cup}E_3\right] &= T(E_{1\cap}E_2)^{\cup}T(E_3) = \left[T(E_1)_{\cap}T(E_2)\right]^{\cup}T(E_3)\\
&= \left[T(E_1)^{\cup}T(E_3)\right]_{\cap}\left[T(E_2)^{\cup}T(E_3)\right]\\
T\left[(E_{1\cap}E_2)^{\cup}E_3\right] &= T(E_{1\cap}E_2)^{\cup}T(E_3) = T\left[(E_1{}^{\cup}E_3)_{\cap}(E_2{}^{\cup}E_3)\right]\\
T\{-\left[(E_{1\cap}E_2)^{\cup}E_3\right]\} &= T\left[-(E_{1\cap}E_2)^{\cup}T(-E_3)\right] = \left[T(-E_1)_{\cap}T(-E_2)\right]_{\cap}T(-E_3)\\
&= \left[T(-E_1)^{\cup}T(-E_2)\right]_{\cap}T(-E_3)\\
T\{-\left[(E_{1\cap}E_2)^{\cup}E_3\right]\} &= T(-E_{1\cap} - E_3)^{\cup}T(-E_{2\cap} - E_3).
\end{aligned}
$$

[17] The demonstration is in de Finetti 1928b: draft #'BD6-02-66', 171: As $T(E_1)_{\cap}T(E_2) :\supset: T(E_1)^{\cup}T(E_2)$

$$
\begin{aligned}
H(E_1{}^{\cup}E_2) &= T(E_1)^{\cup}T(E_2)^{\cup}\left[T(-E_1)_{\cap}T(-E_2)\right] :\supset: T(E_1)^{\cup}T(E_2)^{\cup}T(-E_1)^{\cup}T(-E_2)\\
&= H(E_1)^{\cup}H(E_2)\\
H(E_{1\cap}E_2) &= H(-E_{1\cap} - E_2) :\supset: H(-E_1)^{\cup}H(-E_2) = H(E_1)^{\cup}H(E_2) = H(E_1{}^{\cup}E_2).
\end{aligned}
$$

Thus de Finetti deduces 'the simple formulas':

$$(18)\quad \begin{cases} \dfrac{E_1}{E_3} {}^{\cup} \dfrac{E_2}{E_3} = \dfrac{E_1{}^{\cup}E_2}{E_3} \\[2ex] \dfrac{E_1}{E_3} {}_{\cap} \dfrac{E_2}{E_3} = \dfrac{E_1{}_{\cap}E_2}{E_3} \end{cases}$$

Thus

$$(19)\quad \frac{E_1}{E_2} {}_{\cap} \frac{E_2}{E_3} = \frac{E_1}{E_3}$$

In Part 7 de Finetti shows that an absolute event is a particular subordinate event characterized by any one of the following conditions:

$$(20)\quad \begin{cases} T(E) = E \\ T(-E) = -E \\ H(E) = \oplus \end{cases}$$

that is $E = \frac{E}{\oplus}$. In particular $\oplus = \frac{\oplus}{\oplus}$, and $\ominus = \frac{\ominus}{\oplus}$. As mentioned in 1932:

> We again observe that an absolute event (that we can call, to distinguish it, an event of the type previously considered) can be considered as a particular case of subordinate event (precisely that case in which the 'hypothesis' is a certain event) (de Finetti 1932a: 28, underlined by the author, our translation).

2.5 The Third Truth Value: Insignificant

The third value 'insignificant' is introduced in Part 8. A subordinate event $E = \frac{E_1}{E_2}$ is insignificant (noted $\odot$) when the hypothesis is false ($H(E) = E_2 = \ominus$). In this case the thesis is also false ($T(E) = E_1{}_{\cap}\ominus = \ominus$). Unlike the 'true' and 'false' values it is a 'transitory' value.

$$(21)\quad \begin{cases} \odot = \dfrac{\ominus}{\ominus} \\ T(\odot) = H(\odot) = \ominus \\ -\odot = \odot \\ H(-\odot) = H(\odot) = \ominus \\ T(-\odot) = H(\odot)_{\cap} - T(\odot) = \ominus_{\cap} - \ominus = \ominus \end{cases}$$

Conversely if $-X = X$ then $X = \odot$ ($T(-X) = T(X)$ thus $T(X) = T(X){\cap}T(-X) = \ominus$, $T(-X) = \ominus$, $H(X) = T(X)_{\cap}T(-X) = \ominus$ and $X = \frac{\ominus}{\ominus} = \ominus$). To finish de Finetti gives[18] the relations 22:

[18] The demonstration is in de Finetti 1928c: $T(E^{\cup}\odot) = T(E)$, and $T(-(E^{\cup}\odot)) = T(-E_{\cap}\odot) = \ominus$, thus $H(E^{\cup}\odot) = T(E)$ and $E^{\cup}\odot = \frac{T(E)}{T(E)} = \frac{\oplus}{T(E)}$. Likewise $T(E_{\cap}\odot) = \ominus$ and $T(-(E_{\cap}\odot)) = T(-E^{\cup}\odot) = T(-E)$, thus $H(E_{\cap}\odot) = T(-E)$ and $E_{\cap}\odot = \frac{T(-E)}{T(-E)} = \frac{\ominus}{T(-E)}$.

$$(22)\quad \begin{cases} E^{\cup}\odot = \dfrac{\oplus}{T(E)} \\ E_{\cap}\odot = \dfrac{\ominus}{T(-E)} \end{cases}$$

It is easy[19] to write $E = \frac{E_1}{E_2}$ in a similar manner as the definition of the implication given in 2 (see section 2.1) :

$$(23)\quad E = \frac{T(E)}{H(E)} :=: (T(E)_{\cap}H(E))^{\cup}(\odot_{\cap} - H(E)) :=: (E_{1\cap}E_2)^{\cup}(-E_{2\cap}\odot)$$

This definition corresponds to that of the suppositional connective of Hailperin (see 1996: 36).

These relations allow to clarify the values for the different unary operators (see Table 2).

E	$T(E)$	$H(E)$	$T(-E)$	$H(-E)$	$-T(E)$	$-H(E)$
$\oplus$	$\oplus$	$\oplus$	$\ominus$	$\oplus$	$\ominus$	$\ominus$
$\odot$	$\ominus$	$\ominus$	$\ominus$	$\ominus$	$\oplus$	$\oplus$
$\ominus$	$\ominus$	$\oplus$	$\oplus$	$\oplus$	$\oplus$	$\ominus$

Table 2: Semantic tables for unary operations.

The thesis and hypothesis have been independently rediscovered by several authors. As noted by Mura 2009, the thesis T corresponds to the 'external' connector of Bochvar (1937) 1981. Recently Blamey 2001, Cantwell 2006, Lassiter 2020 introduce both unary operators T and H with different notations. Montagna (2012) sets out three operations which correspond to thesis T ('E is true'), antithesis $-T$ ('E is false') and anti-hypothesis$-H$ ('E is insignificant').

2.6 The Extension to Nested Subordinate Events

Having defined the truth value 'insignificant', de Finetti extends in Part 9 the $\frac{E_1}{E_2}$ subordination operation to the case where E_1 and E_2 are subordinate events with three possible values. He starts by defining T and H.[20]

$$(24)\quad \begin{cases} T\left(\dfrac{E_1}{E_2}\right) = T(E_{1\cap}E_2) = T(E_1)_{\cap}T(E_2) \\ \\ H\left(\dfrac{E_1}{E_2}\right) = T(E_{2\cap}E_1)^{\cup}(T(E_{2\cap} - E_1)) \\ \qquad\qquad = [T(E_2)_{\cap}T(E_1)]^{\cup}[T(E_2)_{\cap}T(-E_1)] \\ \qquad\qquad = T(E_2)_{\cap}[T(E_1)^{\cup}T(-E_1)] = T(E_2)_{\cap}H(E_1) \end{cases}$$

[19] $E = E_{1\cap}E_2 = T(E)$ when $H(E) = \oplus$ and $E = \odot$ when $(-E_2) = -H(E) = \oplus$, thus $E = (T(E)_{\cap}H(E))^{\cup}(\odot_{\cap} - H(E))$.

[20] The relation $T\left(\frac{E_1}{-E_2}\right) = \perp\left(\frac{E_1}{E_2}\right) = T(E_1)_{\cap}T(-E_2)$ is in de Finetti 1928b: draft #'BD6-02-68', 175. These relations will be rediscovered by Montagna 2012.

Thus one can write $\frac{E_1}{E_2}$ in its 'reduced form':

$$(25)\quad \frac{E_1}{E_2} = \frac{T(E_1) \cap T(E_2)}{T(E_2) \cap H(E_1)}$$

And also in the 'subordinate form' by taking $E_1 = \frac{E'_1}{E''_1}$ and $E_2 = \frac{E'_2}{E''_2}$:

$$(26)\quad \frac{E_1}{E_2} = \frac{\frac{E'_1}{E''_1}}{\frac{E'_2}{E''_2}} :=: \frac{E'_1 \cap E''_1 \cap E'_2 \cap E''_2}{E''_1 \cap E'_2 \cap E''_2} :=: \frac{E'_1}{E''_1 \cap E'_2 \cap E''_2}$$

The relation 26, called the 'Import-Export law' in the literature, will appear in appendix of de Finetti (1970) 1975: 328. It entails the following corollaries (Hailperin 1996: 253):

$$(27)\quad \begin{cases} \dfrac{\frac{E'_1}{E''_1}}{E_2} :=: \dfrac{E'_1}{E''_1 \cap E_2} \\ \dfrac{E'_1}{\frac{E''_1}{E_2}} :=: \dfrac{E'_1}{E''_1 \cap E_2} \\ \dfrac{\frac{E'_1}{E''_1}}{\frac{E'_2}{E''_2}} :=: \dfrac{E'_1 \cap E''_1}{E'_2 \cap E''_2} \end{cases}$$

Thus for example for all event E

$$(28)\quad \frac{E}{\odot} = \odot$$

The important consequence of relations 15a, 15b and 26 is that all events E comprising some subordinate events, with the fraction symbol, may be written in a single subordinate form.

2.7 The Level of Probability

De Finetti (1928a: 5) explains that "The logical operations introduced allow the symbolic writing of theorems on the subordinate probabilities", such as for example:

$$(29)\quad P(E) = \frac{P[T(E)]}{P[H(E)]} \quad (\text{with } P[H(E)] \neq 0)$$

De Finetti also gives, with E_1 and E_2 absolute event, the definition of conditional probability and axioms recently rediscovered by some authors (Cantwell 2006, Mura 2009, Rothschild 2014, Lassiter 2020).

$$
(30)\quad \begin{cases}
P(E_1 \cap E_2) & = P(E_2) \times P\left(\frac{E_1}{E_2}\right) \\
P\left(\frac{E_1 \cup E_2}{E_3}\right) & = P\left(\frac{E_1}{E_3}\right) + P\left(\frac{E_2}{E_3}\right) \quad (\text{with } E_1 \cap E_2 \cap E_3 = \ominus) \\
P\left(\frac{E}{E}\right) & = 1 \\
P\left(\frac{-E}{E}\right) & = 0
\end{cases}
$$

The proof of the theorem of conditional probability can be found in de Finetti 1928b: draft #'BD06-02-69', 164 (see the Table 3) and is also discussed in de Finetti 1932a: 29-30.

$$
\begin{array}{cc|ccc}
\frac{E_1}{E_2} & & & E_2 & \\
 & & \oplus & \odot & \ominus \\
\hline
 & \oplus & \oplus 1 & \odot 2 & \odot 3 \\
E_1 & \odot & \odot 4 & \odot 5 & \odot 6 \\
 & \ominus & \ominus 7 & \odot 8 & \odot 9
\end{array}
\qquad
\begin{array}{cc}
+S_1 & (1,2,3) \\
+S_2 & (1,4,7) \\
+S & (1) \\
-pS & (1,7) \\
-pS_1 & (1,2,3,7,8,9) \\
-pS_2 & (1,4,7,3,6,9)
\end{array}
\qquad
\begin{array}{c}
\\
E_1 E_2 \; 1 \\
-E_1 E_2 \; 7 \\
-E_1 - E_2 \; 9 \\
\\
H(E_1)H(E_2) = (1,3,7,9)
\end{array}
$$

$$
\begin{array}{cc|ccc}
 & & S & S_1 & S_2 \\
* & G_1 = & 1-p & 1-P_1 & 1-P_2 \\
 & G_2 = & 0 & 1-P_1 & 0 \\
* & G_3 = & 0 & 1-P_1 & -p_2 \\
 & G_4 = & 0 & 0 & 1-p_2 \\
 & G_5 = & 0 & 0 & 0 \\
 & G_6 = & 0 & 0 & -P_2 \\
* & G_7 = & -p & -p_1 & -P_2 \\
 & G8 = & 0 & -p_1 & 0 \\
* & G9 = & 0 & -p_1 & -p_2
\end{array}
\qquad
\begin{vmatrix}
1-p & 1-P_1 & 1-P_2 \\
-p & -p_1 & -P_2 \\
0 & -p_1 & -p_2
\end{vmatrix}
=
\begin{vmatrix}
0 & 1 & 1 \\
-p & -p_1 & 1-p_2 \\
0 & -p_1 & -p_2
\end{vmatrix}
=
\begin{vmatrix}
0 & 1 & 1 \\
-p & 0 & -p_1 \\
0 & -p_1 & -p_2
\end{vmatrix}
= p_1 \quad -pp_2 \qquad p = \frac{p_1}{p_2}
$$

With $p = P\left(\frac{E_1}{E_2}\right)$, $p_1 = P(E_1)$ and $p_2 = P(E_2)$.

Table 3: De Finetti's proof of theorem 24 (colored in red by the author).

It corresponds to the three first stages of the demonstration in de Finetti (1937: 14) (see section 1.). Here, the matching between stages (i) and (ii) are more detailed. The truth table of subordinate event $E = \frac{E_1}{E_2}$ clarifies the nine possible gains $G_1, \ldots, G_9$ corresponding to nine possible values of E. The gains G_1, G_3, G_7, G_9, marked with an *, correspond to a situation where the conjunction of hypothesis of E_1 and E_2 is true—$H(E_1) \cap H(E_2) = (G_1, G_3, G_7, G_9)$. The bet on $\frac{E_1}{E_2}$ is envisaged only in the case where its constituents E_1 and E_2 are not insignificant (see section 2.8). Now as it is supposed that $E_1 \supset E_2$, the case G_3 should not be considered. The three gains G_1, G_7, G_9, marked with a red *, correspond to the three possible cases E_1, $E_2 \cap - E_1$ and $-E_2$. The coherence constraint implies that the determinant of the linear equation system CP must be null—stage (iii). Since $E_1 \supset E_2$, then $P\left(\frac{E_1}{E_2}\right) = \frac{P(E_1)}{P(E_2)}$. The transition to the general case $E_1 \cap E_2$ in place of E_1 (when $E_1 \supset E_2$) (stage iv) is not mentioned. In de Finetti 1932a: 30 this transition is justified: "the difference is only external, and depends on the fact that when a subordinate event is expressed in its irreducible form (as in the said example), E_1 and $E_1 \cap E_2$ are the same thing".

De Finetti defines the notion of independence:

$$(31)\quad \begin{cases} E_1 \text{ independent of } E_2 :=: P\left(\dfrac{E_1}{E_2}\right) = P(E_1) \\ \text{So} \\ P(E_2) = P\left(\dfrac{E_2}{E_1}\right), \\ P(E_1 \cap E_2) = P(E_1) \times P(E_2) \end{cases}$$

2.8 An Arithmetical Analogy

To finish, in Parts 11 to 14, de Finetti (1928a) introduces a 'remarkable arithmetic analogy'. The event (absolute or subordinate) E, is considered as a random variable x that takes the value $+1$, 0, -1 depending on whether E is true (⊕), insignificant (⊙) or false (⊖). The relation with gains of a conditional bet is obvious even if it is not made explicitly. Each value corresponds to a payoff according to the three possible consequences of a bet on E. If E becomes true, one wins 1€ (2€ – 1€ = 2€$(1-\frac{1}{2}) = S(1-p)$), if E is false, one loses 1€ ($-\frac{1}{2}$2€ $= -pS$) and if E is insignificant one gets back the stake −1€ + 1€ = 0€ $= pS - pS$. Such bet corresponds to the degree of indifference to bet on E or $-E$ equal to $\frac{1}{2}$. This corresponds to Ramsey (1926) 1999's definition of 'ethically neutral proposition'. Thus, the fact that the bettor agrees to bet on E (while he is indifferent between E and $-E$, also amounts to agreeing to bet on the C_i constituents of E that give E true (while the bettor is also indifferent between C_i and $-C_i$). In other words, de Finetti's logic corresponds to a first epistemic level (Baratgin and Politzer 2016), where an individual evaluates the truth or falsity of E (without preference) in the same way a bettor specifies the terms of a bet on the E event in considering only the bi-valued of its constituents (the different possible $(-1, +1)$-model in the semantic table of E reduced to this bi-valued model). A bet is possible if at least a $(-1, +1)$-model gives the value 1. This 'indifferent' step is necessary as a first step in order to elaborate a probability judgment on E (the outlaid pay) which corresponds to the second epistemic level (de Finetti 1980, Baratgin and Politzer 2016).

$$(32)\quad \begin{cases} E = \oplus \;:=:\; x = +1 \\ E = \odot \;:=:\; x = 0 \\ E = \ominus \;:=:\; x = -1 \end{cases}$$

Considering the random variables x_1, x_2 for E_1, E_2, the random variables for $E_1 \cup E_2$ and $E_1 \cap E_2$ correspond to respectively the $\max(x_1, x_2)$ and the $\min(x_1, x_2)$. Hailperin (1996) later formulates the same relations. Hence, it is possible to rearrange in an ascending order the three truth values. Figure 2 represents the three truth values as a function of both the level of knowledge (K) (with ⊙ for the ignorance) and the level of gain (G). As noted by Mura (2009), it is the truth-valued gap interpretation of partial logic (e.g. Blamey 2001).

De Finetti points out that the negation corresponds to the multiplication by -1. $-E$ corresponds to the random variable x':

$$(33)\quad x' = -x$$

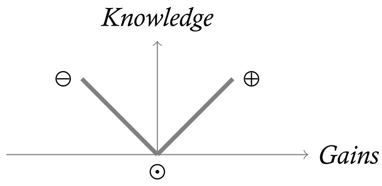


Figure 2: Representation of three truth values according to levels of knowledge and gains.

Let E_1, E_2 be absolute or subordinated events corresponding to random variables x_1, x_2, and $\frac{E_1}{E_2}$ with random variable x, we have:

(34) $x = \dfrac{x_1 x_2 (1 + x_2)}{2}$

Thus if E_2 is false, $x_2 = -1$ and $x = 0$, if E_2 is insignificant, $x_2 = 0$ and $x = 0$. In both cases E is insignificant. Now if E_2 is true, $x_2 = 1$ and $x = x_1$.

Finally the thesis $T(x)$ and hypothesis $H(x)$ are written as:

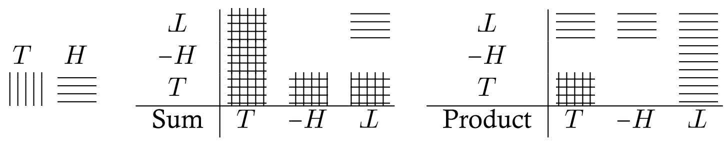
(35) $T(x) = x^2 + x - 1$ and $H(x) = 2x^2 - 1$

3. De Finetti's 2-to-3 Valued Logic

3.1 The First Truth Tables of de Finetti's Trivalent System

Two drafts in de Finetti 1928b explicitly show the tables of truth for negation, product, sum and subordination in the way de Finetti 1936 will present them while de Finetti 1928a, describes them, albeit not explicitly.

The truth tables for the 'sum' and 'product' of tri-events are illustrated in de Finetti 1928b (draft #'BD6-02-66': 171) with as input the thesis T, the anti-hypothesis $-H$ and the antithesis (⅃) (Table 4).



With T thesis, $-H$ anti hypothesis and ⅃ anti thesis.

Table 4: De Finetti's tables for sum and product, following thesis, anti-hypothesis and anti-thesis (from de Finetti 1928b, draft #'BD6-02-66': 171).

By replacing the squared cells ($T = H = \oplus$) by $\oplus$, the empty cells ($H = T = \ominus$) by $\odot$ and the cells with horizontal line ($T = \ominus$ and H=$\oplus$) by $\oplus$ we obtain the tables for the sum and product of subordinate event.

The truth tables for product, sum, subordination and implication connectives (see Table 5)[21] are illustrated in de Finetti (1928b: draft #'BD6-02-58', 163).

[21] For the sake of consistency with section 2.1, E_1 and E_2 are swapped for E_2 and E_1.

$E_1 = E_{1\cap}E_2{}^{\cup}E_{1\cap} - E_2$

$E_{1\cap}E_2$		E_1		
		$\oplus$	$\odot$	$\ominus$
	$\oplus$	$\oplus$	$\odot$	$\ominus$
E_2	$\odot$	$\odot$	$\odot$	$\ominus$
	$\ominus$	$\ominus$	$\ominus$	$\ominus$

$E_1{}^{\cup}E_2$		E_1		
		$\oplus$	$\odot$	$\ominus$
	$\oplus$	$\oplus$	$\oplus$	$\oplus$
E_2	$\odot$	$\oplus$	$\odot$	$\odot$
	$\ominus$	$\oplus$	$\odot$	$\ominus$

$\frac{E_1}{E_2}$		E_1		
$(E_{1\cap}E_2)^{\cup} - E_{2\cap}\odot$		$\oplus$	$\odot$	$\ominus$
	$\oplus$	$\oplus$	$\odot$	$\ominus$
E_2	$\odot$	$\odot$	$\odot$	$\odot$
	$\ominus$	$\odot$	$\odot$	$\odot$

$E_2 \supset_1 E_1$		E_2		
$E_2 \leq_{\odot} E_1$		$\oplus$	$\odot$	$\ominus$
	$\oplus$	$\oplus$	$\oplus$	$\oplus$
E_1	$\odot$	$\ominus$	$\oplus$	$\oplus$
	$\ominus$	$\ominus$	$\ominus$	$\oplus$

(de Finetti 1936: 35)

$E_2 \supset_2 E_1$		E_2		
$(E_{1\cap}E_2)^{\cup} - E_2$		$\oplus$	$\odot$	$\ominus$
	$\oplus$	$\oplus$	$\oplus$	$\oplus$
E_1	$\odot$	$\odot$	$\odot$	$\oplus$
	$\ominus$	$\ominus$	$\odot$	$\oplus$

Table 5: De Finetti's truth tables for product, sum, subordinate, and implication from de Finetti 1928b, draft #'BD6-02-58': 163. In gray, the subordinate and implications definitions.

The truth table for implication is not the one that will be given in de Finetti (1936) (noted respectively '$\supset_1$' and '$\supset_2$'). De Finetti defines it in the same page:

(36) $E_2 \supset_1 E_1 :=: E_{2\cap}E_1 = E_1$

He demonstrates[22] that:

(37) $E_2 \supset_1 E_1 :=: T(E_2) \supset_1 T(E_1)$ and $T(-E_1) \supset_1 T(-E_2)$.

Now setting E_1 and E_2 as $\frac{E_1'}{E_1''}$ and $\frac{E_2'}{E_2''}$ (thus $T(E_1) = E'_{1\cap}E''_1$ and $T(E_2) = E'_{2\cap}E''_2$), relation 37 yields the relation of implication from unconditional events to conditional events discovered by Goodman and Nguyen (1988):

(38) $\dfrac{E_2'}{E_2''} \supset \dfrac{E_1'}{E_1''} :=: E'_{2\cap}E''_2 \supset E'_{1\cap}E''_1$ and $-E'_{1\cap}E''_1 \supset -E'_{2\cap}E''_2$

The implication $\supset_1$ is not equivalent to $-(E_{2\cap} - E_1) :=: E_1{}^{\cup} - E_2 :=: (E_{1\cap}E_2)^{\cup} - E_2$ contrarily to $\supset_2$.[23] Each of these two truth tables corresponds respectively to the generalization to trivalent cases to both definitions 1 and 2 given by de Finetti (1928a) and de Finetti (1932a) (see section 2.1). The implication $\supset_1$ can be defined following the entailment (noted $\leq_{\odot}$) generalizing the bivalued entailment $\leq$ assuming the natural order that $\ominus$ is less true than $\odot$ and $\odot$ is less true than $\oplus$. So $E_2 \leq_{\odot} E_1$ if the value of E_1 is at least as strong as the value of E_2. This order and entailment is supported by some authors (e.g. Milne 1997; Mura 2009; Hailperin 2011; Vidal 2014). The implication $\supset_2$ respects the traditional equivalence to $-(E_{2\cap} - E_1)$.[24] Rescher 1969: 46-52, independently, in order to generalize the bivalent logic system will propose, for the same reason, both successive

[22] if $E_2 \supset_1 E_1$, $E_{2\cap}E_1 = E_1$, thus $T(E_{2\cap}E_1) = T(E_1)$ and also $T(E_{2\cap}E_1) = T(E_2)_{\cap}T(E_1) = T(E_1)$ thus $T(E_2) \supset_1 T(E_1)$. $T(-(E_{2\cap}E_1)) = T(-E_1)$ and $H(E_{2\cap}E_1) = H(E_1)$. Also $T(-(E_{2\cap}E_1)) = T(-E_{2\cap} - E_1) = T(-E_2)_{\cap}T(-E_1) = T(-E_1)$, hence $T(-E_1 \supset_1 -E_2)$.

[23] For the three following cases: $\odot \supset_1 \ominus :=: \ominus$, $\odot \supset_1 \odot :=: \oplus$ and $\ominus \supset_1 \odot :=: \ominus$, while with $\supset_2$, we obtain $\odot$. However $\supset_1$ respects the equivalence $(E_2 \supset_1 E_1)_{\cap}E_2 :=: E_{2\cap}E_1$ while as noted by Égré, Rossi and Sprenger 2020b $(\odot \supset_2 \ominus)_{\cap}E_2 = \odot$ and $\odot_{\cap}\ominus = \ominus$.

[24] It is certainly for this reason that de Finetti modifies the implication table as from 1932.

systems S_3 and K_3 (Kleene' system) which correspond (without the subordinate connective) to both of de Finetti's systems.

3.2 Validity for de Finetti's 2-to-3 Valued Logic

In bi-valued logic, an event E is valid (noted $\therefore E$) if its value is ⊕ under all possible assignments of truth values to its atomic components. An argument E_2 then E_1 is valid (noted $E_2 \therefore E_1$) if it preserves the truth of its premises. That is, if there is no model that renders a premise true and the conclusion false. In the bet analogy, a valid event can be interpreted as a sure bet and the valid inference as a bet preservation it is not possible to bet that E_2 is true without betting that E_1 is true. As it has been pointed out in numerous occasions (see sections 2.1, 2.5, 2.6, 2.7 and 2.8) de Finetti' system corresponds to a logic superimposed on the bivalent logic (i.e a '2-to 3-valued logic'). In generalising the bet analogy, confronted to a subordinate bet, the bettor assigns only the values ⊖ and ⊕ to all the possible atomic components of E (⊖, ⊕)-model in the restricted truth table (the situation where $H(E_1)_\cap H(E_2)$ is true (e.g. see the Tables 1 and 3 restraint to (G_1, G_3, G_7, G_9)). Thus the bettor analyses a 'condensed true table'.[25] Thus

- An event E is ⊙-valid (noted $\therefore_\odot E$) if there is no (⊖, ⊕)-model for which the value in E is ⊖ and if there is at least one model for which its value is ⊕. Concretely if the value column of its semantic restraint table has at least one occurrence of ⊕ and no occurrences of ⊖.
- An argument $E_2 \therefore E_1$ is ⊙-valid (noted $E_2 \therefore_\odot E_1$) if it 'preserves the ⊙-validity'.

This definition of ⊙-validity has been proposed by Hailperin (1996: 35-36 and 246-253), who also took the ⊙-entailment definition Hailperin (2011: 33-34).

The Tables 6 and 7 expose some principles between subordination, implication,[26] ⊙-validity and ⊙-entailment and traditional events and arguments.

The subordination respects the ideal trilemma (Identity, Modus-Ponens and non symmetry) required by Égré, Rossi and Sprenger (2020a). However it does not collapse the implication $\left(E_2 \supset \cancel{E_1 \therefore_\odot} \frac{E_1}{E_2}\right)$ since Supraclassicality fails although all other properties (Import-Export (26), Left Logical Equivalence (10), stronger-than-implication (R2)) are satisfied.

[25] It is therefore very important to consider the de Finetti's system as a 2-to-3 valued logic and not as the traditional interpretation of three valued logic (e.g. Mura 2009, Vidal 2014, Égré, Rossi and Sprenger 2020a). At the probabilistic level, the failure to take into account the restricted form leads to incoherence (see Cantwell 2006).

[26] With "⊃" for bivalent implication.

Hailperin (1996: 248)	R_1 If $\frac{E_1}{E_2}$ is $\odot$-valid then $E_2 \supset E_1$ is valid	
[1]	R_2 Stronger-than-implication	$\frac{E_1}{E_2} \therefore_{\odot} E_2 \supset E_1$
Hailperin (1996: 248)	R_3 Absolute event validity	An absolute event E is $\odot$-valid, if and only if it is valid
[2]	R_5 Entailment versus $\odot$-validity	If $E_2 \leq E_1$ and if there is at least a model that gives E_2 true, then $E_2 \therefore_{\odot} E_1$ and $\therefore_{\odot} \frac{E_1}{E_2}$
[3]	R_6 no $\odot$-validity versus no Entailment	If $E_2 \cancel{\therefore_{\odot}} E_1$, then $E_2 \cancel{\leq_{\odot}} E_1$
[4]	R_7 Conditional Elimination fails	If $\therefore E_2 \cancel{\supset E_1 \text{ then } E_2} \therefore_{\odot} E_1$
Similar to [4]	R_8 Supraclassicality fails	~~If $E_2 \therefore E_1$ then $\therefore_{\odot} \frac{E_1}{E_2}$~~

[1] $E_2 \supset E_1$ is false when $E_2 = \oplus$ and $E_1 = \ominus$. In this case $\frac{E_1}{E_2}$ is also false.
[2] In model where $E_1 = \ominus$ then E_2 also, and in some model where $E_1 = \odot$ then $E_2 = \ominus$ or $E_2 = \odot$. In add there are at least a model that gives E_1 true.
[3] $E_2 \cancel{\therefore_{\odot}} E_1$ if (i) $E_1 = \ominus$ and $E_2 = \odot$ or (ii) $E_1 = \odot$ and $E_2 = \ominus$. in these both cases $E_2 \cancel{\leq_{\odot}} E_1$.
[4] Take $E_2' = \ominus$ and $E_1' = \oplus$.

Table 6: Relations between implication and subordination connectives.

Proofs	**Events**	**Epistemic belief level**		**Epistemic degree of belief level**[1]
Hailperin (1996: 249)[3]	Identity	$\therefore_{\odot} \frac{A_1}{A_1}$		$P\left(\frac{A_1}{A_1}\right) \in [0,1]$
Hailperin (1996: 249)		$\therefore_{\odot} \frac{A_1 \cup A_2}{A_1}$		$P\left(\frac{A_1 \cup A_2}{A_1}\right) \in [0,1]$
Hailperin (1996: 249)		$\therefore_{\odot} \frac{A_1}{A_1 \cap A_2}$		$P\left(\frac{A_1}{A_1 \cap A_2}\right) \in [0,1]$
Hailperin (1996: 249)		$\cancel{\therefore_{\odot}} \frac{E}{A_1 \cap -A_1}$		$P\left(\frac{C}{A \cap -A}\right) \in [0,1]$
	Arguments			
R_3	And-introduction	$E_2, E_1 \therefore_{\odot} E_2 \cap E_1$	$\leq_{\odot}$	$P(E_2 \cap E_1) \in [\max\{0, P(E_2) + P(E_1) - 1\}, \min\{P(E_2), P(E_1)\}]$
R_3	And-elimination	$E_2 \cap E_1 \therefore_{\odot} E_2$	$\leq_{\odot}$	$P(E_2) \in [P(E_2 \cap E_1), 1]$
R_3	Or-introduction	$E_2, E_1 \therefore_{\odot} E_2 \cup E_1$	$\leq_{\odot}$	$P(E_2 \cup E_1) \in [\max\{P(E_2), P(E_1)\}, \min\{P(E_2) + P(E_1), 1\}]$
R_3		$E_2 \therefore_{\odot} E_2 \cup E_1$	$\leq_{\odot}$	$P(E_2 \cup E_1) \in [P(E_2), 1]$
R_3	If-introduction	$E_2, E_1 \therefore_{\odot} \frac{E_1}{E_2}$	$\leq_{\odot}$	$P\left(\frac{E_1}{E_2}\right) \in \left[\max\left\{0, \frac{(P(E_1)+P(E_2)-1)}{P(E_2)}\right\}, \min\left\{\frac{P(E_1)}{P(E_2)}, 1\right\}\right]$
[3]	Consequent to 'if'	$E_1 \therefore_{\odot} \frac{E_1}{E_2}$	$\cancel{\leq_{\odot}}$	$P\left(\frac{E_1}{E_2}\right) \in [0,1]$
Verification of $\leq_{\odot}$ and R_5	'And' to 'if'	$E_2 \cap E_1 \therefore_{\odot} \frac{E_1}{E_2}$	$\leq_{\odot}$	$P\left(\frac{E_1}{E_2}\right) \in [P(E_2 \cap E_1), 1]$
Similar to [3]	'Or' to 'if not'	$E_2 \cup E_1 \therefore \frac{E_1}{-E_2}$	$\cancel{\leq_{\odot}}$	$P\left(\frac{E_1}{-E_2}\right) \in [0, P(E_2 \cup E_1)]$
Hailperin (1996: 248)	Modus Ponens	$\frac{E_1}{E_2}, E_2 \therefore_{\odot} E_1$	$\leq_{\odot}$	$P(E_1) \in \left[P(E_2) \times P\left(\frac{E_1}{E_2}\right), 1 + P(E_2)\left(P\left(\frac{E_1}{E_2}\right) - 1\right)\right]$
Similar to [3]	Denying the Antecedent	$\frac{E_1}{E_2}, -E_2 \therefore_{\odot} -E_1$	$\cancel{\leq_{\odot}}$	$P(-E_1) \in \left[\left(1 - P\left(\frac{E_1}{E_2}\right)\right) \times (1 - P(-E_2)), 1 - P\left(\frac{E_1}{E_2}\right) \times (1 - P(-E_2))\right]$
R_2 and R_3	Modus Tollens	$\frac{E_1}{E_2}, -E_1 \therefore_{\odot} -E_2$	$\leq_{\odot}$	$P(-E_2) \in \left[\max\left\{\frac{1-P\left(\frac{E_1}{E_2}\right)-P(-E_1)}{1-P\left(\frac{E_1}{E_2}\right)}, \frac{P\left(\frac{E_1}{E_2}\right)+P(-E_1)-1}{P\left(\frac{E_1}{E_2}\right)}\right\}, 1\right]$
Similar to [3]	Affirming the Consequent	$\frac{E_1}{E_2}, E_1 \therefore_{\odot} E_2$	$\cancel{\leq_{\odot}}$	$P(E_2) \in \left[0, \min\left\{\frac{P(E_1)}{P\left(\frac{E_1}{E_2}\right)} \frac{1-P(E_1)}{1-P\left(\frac{E_1}{E_2}\right)}\right\}\right]$
Hailperin (1996: 248)	Hypothetical syllogism	$\frac{B}{E_2}, \frac{E_1}{B} \therefore_{\odot} \frac{E_1}{E_2}$	$\leq_{\odot}$	$P\left(\frac{E_1}{E_2}\right) \in [0,1]$
Hailperin (2011: 34) & R_5	Cut	$\frac{E_2}{E_3}, \frac{E_1}{E_3 \cap E_2} \therefore_{\odot} \frac{E_1}{E_3}$	$\leq_{\odot}$	$P\left(\frac{E_1}{E_3}\right) \in \left[P\left(\frac{E_2}{E_3}\right) \times P\left(\frac{E_1}{E_3 \cap E_2}\right), P\left(\frac{E_2}{E_3}\right) \times P\left(\frac{E_1}{E_3 \cap E_2}\right) + 1 - P\left(\frac{E_2}{E_1}\right)\right]$
Hailperin (1996: 248)	Proofs by cases	$\frac{E_1}{E_2}, \frac{E_1}{-E_2} \therefore_{\odot} E_1$	$\cancel{\leq_{\odot}}$	$P(E_1) \in \left[\min\left\{P\left(\frac{E_1}{E_2}\right), P\left(\frac{E_1}{-E_2}\right)\right\}, \max\left\{P\left(\frac{E_1}{E_2}\right), P\left(\frac{E_1}{-E_2}\right)\right\}\right]$
Hailperin (1996: 248)	Reductio ad absurdum	$\frac{E_1}{E_2}, \frac{-E_1}{E_2} \therefore_{\odot} -E_2$	$\leq_{\odot}$	$P(-E_2) \in [0,1]$
Hailperin (2011: 34) & R_5	Cautious monotonicity	$\frac{E_2}{E_3}, \frac{E_1}{E_3} \therefore_{\odot} \frac{E_1}{E_3 \cap E_2}$	$\leq_{\odot}$	$P\left(\frac{E_1}{E_3 \cap E_2}\right) \in \left[\max\left\{0, \frac{P\left(\frac{E_2}{E_3}\right)+P\left(\frac{E_1}{E_3}\right)-1}{P\left(\frac{E_2}{E_3}\right)}\right\}, \min\left\{\frac{P\left(\frac{E_1}{E_3}\right)}{P\left(\frac{E_2}{E_3}\right)}, 1\right\}\right]$
Vidal (2014) & R_5	'Switches'	$\frac{E_1}{E_3 \cap E_2}, \therefore_{\odot} \frac{E_1}{E_3} \cup \frac{E_1}{E_2}$	$\leq_{\odot}$	$P\left(\frac{E_1}{E_3} \cup \frac{E_1}{E_2}\right) \in \left[P\left(\frac{E_1}{E_3 \cap E_2}\right), 1\right]$
[4]	'Not-E_2' to 'if'	$-E_2 \cancel{\therefore_{\odot}} \frac{E_1}{E_2}$	$\cancel{\leq_{\odot}}$	$P\left(\frac{E_1}{E_2}\right) \in [0,1]$
Similar to [4]	Symmetry	$\frac{E_1}{E_2} \cancel{\therefore_{\odot}} \frac{E_2}{E_1}$	$\cancel{\leq_{\odot}}$	$P\left(\frac{E_2}{E_1}\right) \in [0,1]$
Hailperin (1996: 249)	Contraposition	$\frac{E_1}{E_2} \cancel{\therefore_{\odot}} \frac{-E_2}{-E_1}$	$\cancel{\leq_{\odot}}$	$P\left(\frac{-E_2}{-E_1}\right) \in [0,1]$
Similar to [4]	Strengthening	$\frac{E_1}{E_2}, \cancel{\therefore_{\odot}} \frac{E_1}{E_2 \cap B}$	$\cancel{\leq_{\odot}}$	$P\left(\frac{E_1}{E_2 \cap B}\right) \in [0,1]$

[1] Probability interval are found with water tank analogy of Politzer 2016.
[2] As underlined by Hailperin (1996: 249), $\odot$-validity is not preserved under substitution. e.g. $\frac{A_1 \cap -A_1}{A_1 \cap -A_1}$ is not $\odot$-valid.
[3] When $A = \ominus$, then $\frac{E_1}{E_2} = \odot$ thus $E_1 \therefore_{\odot} \frac{E_1}{E_2}$. However when $A = \ominus$ and $C = \oplus$, $\frac{E_1}{E_2} \cap C = \odot$.
[4] When $\frac{E_1}{E_2} = \odot$, $E_2 \supset E_2 :=: \oplus$.

Table 7: Main Arguments following their $\odot$validity and their probability interval.

4. Conclusion: Primacy of De Finetti's Concepts

Milne (2012) believes that Joseph Schächter was the first author in 1935 to propose Table 1 for indicative conditional *If* E_2, E_1. Recently, Égré, Rossi and Sprenger (2020a) wonder whether the primacy of the truth table might not belong to Hans Reichenbach. Indeed, Reichenbach (1935) 1949: 400 and Reichenbach 1935: 42present a truth table with three values that the author notes '1', '0' and '?' for 'probabilistic implication' $E_2 \ni\!\!- E_1$[27] in the specific 'limiting cases' where

[27] Introduced as early as 1925 (see Reichenbach [1925] 1978: 89-90).

probabilities of E_2 and E_1 are 0 or 1. [28] In the general case, the truth table of the "probabilistic implication" corresponds to a plurivalent logic where the truth values correspond to the numerical values of the degrees of probability. In 1935, de Finetti makes a critical review of this point (see de Finetti 1927-1935: 62-65 and the correspondence 50-61 to which, Reichenbach responds opposing the 2-to-3-valued-logic approach of de Finetti). [29] De Finetti (1936) will specify that this infinite value logic can be reduced to his 2-to-3 valued logic (abandoning Reichenbach's frequentist presuppositions to establish probabilities). It was not until 1941 that Reichenbach presented his three-value quantum logic in a form similar to de Finetti's, but with a different implication (Reichenbach 1944, de Finetti (1970) 1975).

As carefully established by this paper we support that as early as 1928, Bruno de Finetti had expressed the idea of the table for the conditional and has already conceptualized the whole logic of tri-events.[30]

References

Baratgin, J., Over, D.E. and Politzer, G. 2013, "Uncertainty and the De Finetti Tables", *Thinking & Reasoning*, 19, 308-28.

Baratgin, J., Over, D.E. and Politzer, G. 2014, "New Psychological Paradigm for Conditionals and General de Finetti Tables", *Mind & Language*, 29, 73-84.

Baratgin, J. and Politzer, G. 2016, "Logic, Probability and Inference: A Methodology for a New Paradigm", in Macchi, L., Bagassi, M. and Viale, R. (eds.), *Cognitive Unconscious and Human Rationality*, Cambridge, MA: The MIT Press, 119-42.

Baratgin, J., Politzer, G., Over, D.E. et al. 2018, "The Psychology of Uncertainty and Three-Valued Truth Tables", *Frontiers in Psychology*, 9, 1479.

Bassetti, F. and Regazzini, E. 2008, "The Unsung de Finetti's First Paper about Exchangeability", *Rendiconti di Matematica*, 28, 1-17.

Beaver, D.I. and Krahmer, E. 2001, "A Partial Account of Presupposition Projection", *Journal of Logic, Language and Information*, 10, 147-82.

Blamey, S.R. 2001, "Partial logic", in Gabbay, D. and Günthner, F. (eds.), *Handbook of Philosophical Logic*, Vol. V, Amsterdam: Elsevier Science Publishers, 261-353.

Bochvar, D.A. (1937) 1981, "On a Three-Valued Logical Calculus and Its Application to the Analysis of the Paradoxes of the Classical Extended Functional Calculus", trans. by Bergmann, M., *History and Philosophy of Logic*, 2, 87-112.

[28] The general case being formulated by an inequality close to the interval of the "introduction of the conditional" (see Table 7): $P\left(\frac{E_1}{E_2}\right)$ (noted "u") $\in \left[\frac{(P(E_1)+P(E_2)-1)}{P(E_2)}, \frac{P(E_1)}{P(E_2)}\right]$. The three values "1", "0" and "?" represent respectively the values of u for $P(E_1) = P(E_2) = 1$, $P(E_1) = 0$ and $P(E_2) = 1$, and $P(E_2) = 0$.

[29] The text can also be found at http://www.brunodefinetti.it/Opere/Rec%20B.de%20Finetti-Hans%20Reichenbach.pdf.

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Cantwell, J. 2006, "The Laws of Non-bivalent Probability", *Logic and Logical Philosophy*, 15, 163-71.

Cifarelli, D.M. and Regazzini, E. 1996, "De Finetti's Contribution to Probability and Statistics", *Statistical Science*, 11, 253-82.

Coletti, G. and Scozzafava, R. 2002, *Probabilistic Logic in a Coherent Setting*, Trends in Logic, Dordrecht: Kluwer.

de Finetti, B. 1927, "Probabilità che il massimo comune divisore di n numeri scelti ad arbitrio sia un numero dato", *Rendiconti del R. Istituto Lombardo di Scienze e Lettere*, 60, 3-8.

de Finetti, B. 1927-1935, *Logica plurivalente (1927-1935)*, de Finetti Papers, III. Research, 1927-2000 (Box 6, Folder 2), Pittsburgh: University of Pittsburgh (ULS Digital Collections), URL: https://digital.library.pitt.edu/islandora/object/pitt%A31735033466552/viewer.

de Finetti, B. 1928a, "L'EVENTO SUBORDINATO come ente logico", in de Finetti 1927-1935, 154-60.

de Finetti, B. 1928b, "de Finetti's Drafts", in de Finetti 1927-1935, 161-95.

de Finetti, B. 1928c, "Logica degli eventi", in de Finetti 1927-1935, 173.

de Finetti, B. 1928d, "Sulle probabilità numerabili e geometriche", *Rendiconti del R. Istituto Lombardo di Scienze e Lettere*, 61, 817-24.

de Finetti, B. 1930, "Fondamenti logici del ragionamento probabilistico", *Bollettino dell'Unione Matematica Italiana*, 9, 258-61.

de Finetti, B. 1931, "Sul significato soggettivo della probabilità", *Fundamenta Mathematicae*, 17, 298-329.

de Finetti, B. 1932a, "Lezioni sulla probabilità", in *Bruno de Finetti Papers*, III. Research, 1927-2000 (Box 5, Folder 10), Pittsburgh: University of Pittsburgh (ULS Digital Collections), URL: https://digital.library.pitt.edu/islandora/object/pitt%3A31735033466297/viewer.

de Finetti, B. 1932b, "Sulla legge di probabilità degli estremi", *Metron*, 9, 127-38.

de Finetti, B. 1933, "Sul concetto di probabilità", *Rivista ltaliana di Statistica Economia, e Finanza*, 5, 723-47.

de Finetti, B. 1936, "La logique de la probabilité", *Actualités Scientifiques et Industrielle*, 391, 31-9, English translation by Angell, R.B., "The Logic of Probability", *Philosophical studies*, Vol. 77, 1995, 181-90.

de Finetti, B. 1937, "La Prévision: Ses lois logiques, ses sources subjectives", *Annales de l'Institut Henri Poincaré*, 7, 1-68, English translation by Kyburg, H.E. Jr. with new notes added by the author, "Foresight: Its Logical Laws, Its Subjective Sources", in Kyburg, H.E. Jr. and Smokler, H.E. (eds.), *Studies in Subjective Probability*, New York: Wiley, 1964, 53–118.

de Finetti, B. 1967, "Sur quelques conventions qui semblent utiles", *Revue Roumaine de Mathematiques Pures et Appliquées*, 12, 1227-33.

de Finetti, B. (1970) 1974, *Theory of Probability: A Critical Introductory Treatment*, trans. by Machi, A. and Smith, A., Vol. 1, Classics Library, New York: Wiley, originally published in *Teoria delle probabilità: Sintesi introduttiva con appendice critica*, Torino: Einaudi, 1-347.

de Finetti, B. (1970) 1975, *Theory of Probability: A Critical Introductory Treatment*, trans. by Machi, A. and Smith, A., Vol. 2, Classics Library, New York: Wiley, originally published in *Teoria delle probabilità: Sintesi introduttiva con appendice critica*, Torino: Einaudi, 349-739.

de Finetti, B. 1980, "Probabilità", in *Enciclopedia*, Vol. X, Torino: Einaudi, 1146-87.

de Finetti, B. (1934) 2006, *L'invenzione della verità*, Milano: Cortina.

de Finetti, B. (1979) 2008, "Prevision, Random Quantities, and Trievents", in *Philosophical Lectures on Probability*, ed. by Mura, A., trans. by Hosni, H., Vol. 340, Synthese Library, Dordrecht: Springer, 175-85, originally published as "La probabilità come prezzo", in Id. *Filosofia della probabilità*, ed. by Mura, A., Milano: Il Saggiatore, 253-62.

de Finetti, F. and Nicotra, L. 2008, *Bruno de Finetti: Un matematico scomodo*, Livorno: Belforte.

Delli Rocili, L. and Maturo, A. 2013, "Logica del certo e dell'incerto per la scuola primaria", *Science & Philosophy*, 1, 37-58.

Douven, I. 2016, "On de Finetti on Iterated Conditionals", in Beierle, C., Brewka, G. and Thimm, M. (eds.), *Computational Models of Rationality: Essays Dedicated to Gabriele Kern-Isberner on the Occasion of Her 60th Birthday*, London: College Publications, 265-79.

Dubois, D. and Prade, H. 1994, "Conditional Objects as Nonmonotonic Consequence Relationships", *IEEE Transactions on Systems, Man and Cybernetics*, 24, 12, 1724-40.

Ducrot, O. 1969, "Présupposés et sous-entendus", *Langue française*, 4, 30-43, URL: https://www.persee.fr/doc/lfr_0023-8368_1969_num_4_1_5456.

Ducrot, O. (1980) 2008, *Dire et ne pas dire: principes de sémantique linguistique*, Collection Savoir, Paris: Hermann.

Égré, P., Rossi, L. and Sprenger, J. 2020a, "De Finettian Logics of Indicative Conditionals – Part 1: Trivalent Semantics and Validity", *Journal of Philosophical Logic*, 50, 187-213.

Égré, P., Rossi, L. and Sprenger, J. 2020b, "Gibbardian Collapse and Trivalent Conditionals", in Kaufmann, S., Over, D. and Sharma, G. (eds.), *Conditionals: Logic, Linguistics and Psychology*, Cham: Palgrave MacMillan.

Galavotti, M.C. 2018, "The Sessions on Induction and Probability at the 1935 Paris Congress: An Overview", *Philosophia Scientiæ*, 22-3, 3, 213-32, URL: https://www.cairn.info/revue-philosophia-scientiae-2018-3-page-213.htm.

Goodman, I. and Nguyen, H. 1988, "Conditional Objects and the Modelling of Uncertainties", in Gupta, M. and Yamakawa, T. (eds.), *Fuzzy computing: Theory, Hardware, and Applications*, Amsterdam: North-Holland, 119-38.

Goodman, I., Nguyen, H. and Walker, E. 1991, *Conditional Inference and Logic for Intelligent Systems*, Amsterdam: North-Holland, I-VIII, 1-288.

Hailperin, T. 1996, *Sentential Probability Logic: Origins, Development, Current Status, and Technical Applications*, Bethlehem, PA: Lehigh University Press.

Hailperin, T. 2011, *Logic with a Probability Semantics*, Lanham, MD (USA): Rowman & Littlefield Publishing Group, Incorporated.

Lassiter, D. and Baratgin, J. 2021, "Nested Conditionals and Genericity in the de Finetti Semantics", *Thought: A Journal of Philosophy*, 10, 42-52.

Lassiter, D. 2020, "What We Can Learn from How Trivalent Conditionals Avoid Triviality", *Inquiry: An Interdisciplinary Journal of Philosophy*, 63, 1087-114.

Milne, P. 1997, "Bruno de Finetti and the Logic of Conditional Events", *The British Journal for the Philosophy of Science*, 48, 195-232.

Milne, P. 2012, "Indicative Conditionals, Conditional Probabilities, and the Defective Truth-Table?: A Request for More Experiments", *Thinking & Reasoning*, 18, 196-224.

Montagna, F. 2012, "Partially Undetermined Many-Valued Events and Their Conditional Probability", *Journal of Philosophical Logic*, 41, 3, 563-93.

Morini, S. 2007, "Bruno de Finetti: l'origine de son subjectivisme", *Electronic Journal for History of Probability and Statistics*, 3.

Mura, A. 2009, "Probability and the Logic of de Finetti's Trievents", in Galavotti, M. (ed.), *Bruno de Finetti Radical Probabilist*, London: College Publications, 201-42.

Nakamura, H., Shao, J., Baratgin, J. et al. 2018, "Understanding Conditionals in the East: A Replication Study of Politzer et al. (2010) With Easterners", *Frontiers in Psychology*, 9, 505, URL: https://www.frontiersin.org/article/10.3389/fpsyg.2018.00505.

Politzer, G., Jamet, F. and Baratgin, J. 2020, "Children's Comprehension of Conditional Requests", in Elqayam, S., Douven, I., Evans, J. et al. (eds.), *Logic and Uncertainty in the Human Mind: A Tribute to David E. Over*, London: Routledge, 161-77.

Politzer, G. 2016, "Deductive Reasoning under Uncertainty: A Water Tank Analogy", *Erkenntnis*, 81, 3, 479-506.

Ramsey, F. (1926) 1999, "Truth and Probability", in Mellor, D. (ed.), *Philosophical Papers*, Cambridge: Cambridge University Press, 52-94.

Regazzini, E. 2013, "The Origins of de Finetti's Critique of Countable Additivity", in Jones, G. and Shen, X. (eds.), *Advances in Modern Statistical Theory and Applications: A Festschrift in honor of Morris L. Eaton*, Vol. 10, Collections, Beachwood, OH (USA): Institute of Mathematical Statistics, 63-82.

Reichenbach, H. 1935, "Wahrscheinlichkeitslogik", *Erkenntnis*, 5, 37-43.

Reichenbach, H. 1944, *Philosophical Foundations of Quantum Mechanics*, Berkeley and Los Angeles: University of California Press.

Reichenbach, H. (1935) 1949, *The Theory of Probability*, trans. by Hutten, E. and Reichenbach, M., Berkeley: University of California Press, originally published as *Wahrscheinlichkeitslehre*, Leiden: Sijthoff.

Reichenbach, H. (1925) 1978, "The Causal Structure of the World and the Difference between Past and Future", in Reichenbach, M. and Cohen, R.S. (eds.), *Hans Reichenbach Selected Writings 1909–1953*, Vol. 2, Vienna Circle Collection, Dordrecht: Reidel, 81-119.

Rescher, N. 1969, *Many-Valued Logic*, New York: McGraw-Hill.

Rothschild, D. 2014, "Capturing the Relationship between Conditionals and Conditional Probability with a Trivalent Semantics", *Journal of Applied Non-Classical Logics*, 24, 1-2, 144-52.

Vidal, M. 2014, "The Defective Conditional in Mathematics", *Journal of Applied Non-Classical Logics*, 24, 1-2, 169-79.

Bypassing Lewis' Triviality Results. A Kripke-Style Partial Semantics for Compounds of Adams' Conditionals

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Abstract

According to Lewis' Triviality Results (LTR), conditionals cannot satisfy the equation (E) $\mathbf{P}(C\ if\ A) = \mathbf{P}(C \mid A)$, except in trivial cases. Ernst Adams (1975), however, provided a probabilistic semantics for the so-called *simple conditionals* that also satisfies equation (E) and provides a probabilistic counterpart of logical consequence (called *p*-entailment). Adams' probabilistic semantics is coextensive to Stalnaker-Thomason's (1970) and Lewis' (1973) semantics as far as simple conditionals are concerned. A theorem, proved in McGee 1981, shows that no truth-functional many-valued logic allows a relation of logical consequence coextensive with Adams' *p*-entailment.

This paper presents a modified modal (Kripke-style) version of de Finetti's semantics that escapes McGee's result and provides a general truth-conditional semantics for indicative conditionals. It agrees with Adams' logic and is not affected by LTR. The new framework encompasses and extends Adams' probabilistic semantics (APS) to compounds of conditionals. A generalised set of axioms for probability over the set of tri-events is provided, which coincide with the standard axioms over the set of the two-valued ordinary sentences.

Keywords: Conditionals, Probability logic, de Finetti, Tri-events, Adams' logic, Stalnaker's thesis, Partial logic, Lewis' triviality results, Ramsey test.

1. Introduction: de Finetti's Tri-events

According to the so-called Stalnaker's Thesis,[1] the probability $\mathbf{P}(p \Rightarrow q)$ of an indicative conditional "if p then q" equals the probability $\mathbf{P}(q \mid p)$ of the consequent given the antecedent. However, according to so-called Lewis' Triviality Results

[1] I use the expression 'Stalnaker's Thesis' because Stalnaker 1970 prompted its consideration among philosophers, and it is often so-called in the literature on conditionals. As a matter of fact, the idea goes back to Ramsey (1928) 1990 and also to de Finetti (1934) 2006.

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(Lewis 1976), hereafter abbreviated by 'LTR', there is no connective "$\Rightarrow$" that satisfies, in general, the equation $\mathbf{P}(p \Rightarrow q) = \mathbf{P}(q \mid p)$, except in trivial cases. These results seem to support either the view, maintained by several scholars, according to which conditionals lack in general truth conditions or the view that the equation is incorrect, at least concerning non-simple conditionals.

LTR (on all its versions, including Hájek 1989, that appears to be one version that limits the set of assumptions to the non-dispensable ones) rest on their premises. Among them, two appear to be crucial:

1. Conditional sentences express two-valued propositions, so that n $(0 < n < \omega)$ conditional sentences generate a Boolean algebra with at most 2^n elements (up to logical equivalence);

2. the laws of finite probability hold for conditionals as they are. No modification is required.

Condition (2) is natural in the presence of the condition (1), while denial of condition (1) not necessarily entails the denial of condition (2). What if we drop both these two premises? Bruno de Finetti ([1934] 2006, [1935] 1995) proposed a new framework, where (a) conditionals (called *tri-events*) have partial truth conditions and (b) probabilities are defined over a lattice which, if genuine tri-events are involved, is not a Boolean algebra. [2] In de Finetti's approach, the probability of ordinary sentences follows the common laws of finite probability, while the probability of tri-events follows slightly more general laws, always consistently satisfying the equation $\mathbf{P}(p \Rightarrow q) = \mathbf{P}(q \mid p)$. Moreover, when p and q are ordinary two-valued sentences, it holds that if $\mathbf{P}(p) > 0$, $\mathbf{P}(p \Rightarrow q) = \frac{\mathbf{P}(p \wedge q)}{\mathbf{P}(p)}$, so that in the case of simple conditionals (that is in the case of those conditionals in which both the antecedent and the consequent express ordinary two-valued propositions), conditional probability follows the usual ratio formula.

De Finetti did not define the relation of logical consequence between tri-events. By contrast, Ernest Adams (1975) developed a probabilistic semantics for simple conditional sentences with no impossible antecedent (hereafter abbreviated by 'APS'). He defined for such conditionals a logical consequence relation in probabilistic terms (called *p*-entailment). Adams *p*-entailment coincides with the standard logical consequence when only ordinary sentences are involved. Moreover, *p*-entailment is in excellent agreement with intuitions in most cases. It turns out (Adams 1977, see also Gibbard 1981) that, as far as simple conditionals with no impossible antecedent are concerned, the logical consequence relations defined respectively in the Stalnaker-Thomason semantics (1970) and the Lewis' semantics (1973) are coextensive to *p*-entailment.

According to Adams, conditionals lack truth conditions except in the case in which they come down to ordinary sentences. For Adams, *p*-entailment belongs

[2] Recently, J. Baratgin (2021) showed that de Finetti developed in great detail in his unpublished manuscripts held at the University of Pittsburgh (1927-1935) his theory of tri-events as early as 1928. For very recent discussions and development of definettian logics with respect to its three-valued alternatives see Égré, Rossi and Sprenger 2021a, 2021b, Lassiter and Baratgin 2021. One can find a previous account of definettian logics in Milne 2004.

to a probabilistic semantics in which the correctness of an inference cannot be presented as some genuine preservation property concerning the truth conditions of the premises and the conclusion. However, to say that *p*-entailment preserves high probability is somewhat misleading: when the number of the premises is sufficiently large, the probability of each premise can be very high (short of full certainty),[3] while the probability of a *p*-entailed conclusion very low. Moreover, as Kleiter (2018) points out, there are inferences (like affirming the consequent) that are not *p*-valid, while "allow to constrain the probabilities of conclusions in the same way as the modus ponens or the modus tollens", and are such that, from the viewpoint of a probabilistic logic without truth conditions, should not be considered as fallacious. To vindicate Adams' probability logic we need a parallel truth-conditional logic. And extending Adams' probability logic to compounds of conditionals requires a new truth-conditional semantics.

Is it possible to equip de Finetti's logic of tri-events with a logical consequence relation that agrees with Adams *p*-entailment? Due to a theorem by McGee (1981), this is not the case (on this point see also Adams 1995, Schulz 2009).

A closer analysis of de Finetti's logic shows that, concerning simple conditionals with no impossible antecedents, it differs from Adams' logic because every sentence of the form $\varphi \Rightarrow \varphi$, while is probabilistically valid in Adams' logic, is not valid in de Finetti's logic. Indeed, while it may not be false, it may well not be true.

This consideration motivates the present effort to provide a new semantics for tri-events such that every sentence of the form $\varphi \Rightarrow \varphi$ turns out to be true whenever φ does express a proposition that may be true. Since truth depends here on a modal condition, to express this fact by a schema, we need a language with modal operators.

This paper presents such a semantical account in a Kripke-style manner. We will define probability axioms and probability functions that satisfy, in a general way, the equation $\mathbf{P}(p \Rightarrow q) = \mathbf{P}(q \mid p)$. We can define both notions of *p*-entailment and truth-conditional logical consequence in this theory. It turns out that our *p*-entailment encompasses Adams *p*-entailment since it coincides with it concerning simple conditionals. Moreover, a relation of logical consequence that is coextensive with general *p*-entailment is definable in truth-conditional terms. Like Adams

[3] McGee (1994) has refined Adams' *p*-entailment by the so-called Popper probability functions. McGee defines the notion of *strict entailment* (renamed in Adams 1998: 152, *weak validity* after recognition that it is not stricter but weaker than *p*-entailment), that preserves probability 1. McGee approach has the valuable advantage that strict entailment is compact so that it needs no restriction to finite sets of premises. Despite that, we stick to the original Adams' 1975 approach. Following Adams 1975: 49-50, we require that in probabilistic arguments, probability assignments are *proper* for $(\psi \mid \varphi)$ if $\mathbf{P}(\varphi) > 0$ and that only proper assignments occur in probabilistic arguments. Our truth-conditional semantics fits this notion of *p*-entailment. The main reason for this choice is that when $\mathbf{P}(\varphi) = 0$ and φ is a contingent sentence, the probability that $(\psi \mid \varphi)$ lacks a truth-value is 1 in the present approach. Since we consider probability as probability of truth, it does not make sense (at least as far as inferential reasoning is concerned) to assign a definite probability value to sentences that, with probability 1, lack a truth-value. Further considerations will be made in section 6.

probabilistic semantics, we confine the present truth-conditional semantics to the so-called *indicative conditionals*.[4]

The present theory provides a new semantical account for indicative conditionals, which is alternative to other semantics (like Stalnaker-Thomason's one) as far as compounds of indicative conditionals are concerned. I discuss the problem whether this semantics is better suited in dealing with compounds of conditionals in the final part of the paper.

While de Finetti in (1935) 1995 presented his logic as a three-valued logic, actually he meant it as a *partial* logic. Indeed, he meant the third value out of true and false not as *undetermined* (that is as unknown or unknowable), but as *null*, that is a genuine truth-value gap. De Finetti maintained throughout his life (see de Finetti [1979] 2008: 169) that we may interpret '|' in a probability statement of the form $\mathbf{P}(q \mid p)$ as a logical connective and that a sentence of the form $(q \mid p)$ may be interpreted as 'q supposing that p', according to the following truth-table:

Conditioning
$q \mid p$

			q	
		1	0	u
	1	1	0	u
p	0	u	u	u
	u	u	u	u

When p and q are ordinary events, a tri-event $(q \mid p)$ is considered true if both p and q are true, false if p is true while q is false and null (i.e. neither true nor false) if p is false.

De Finetti did not aim at solving philosophical problems about conditionals, as logicians and philosophers of language today understand them. Instead, his problem was to extend to conditional probability a well-known fact that holds for absolute (finite) probability: that the laws of probability force the probability of a proposition p to be 1 (or 0) if and only if p is logically true (or logically false). Since the laws of finite probability force also, in special cases, *conditional* probabilities or compound of conditional probabilities to be 0 or 1, the problem arises whether also in this case these extreme probability values correspond to truth-values. This approach would require an extension of Boolean logic to include conditional events. This requirement explains why de Finetti named his theory of tri-events 'the logic of probability'.

For example, the following equation holds for conditional probability:

(1) $\mathbf{P}(q \mid p) + \mathbf{P}(\neg q \mid p) = 1$

provided p is possibly true.[5] In the case of absolute probability the equation $\mathbf{P}(q) + \mathbf{P}(\neg q) = 1$ is linked to the logical fact that either q or $\neg q$ is true. Can we interpret

[4] In my paper Mura 2016, I have outlined an extension of the present theory, covering counterfactual conditionals as well. This extension will not be considered here.

[5] de Finetti 1936 anticipates (without providing an axiomatic treatment) the idea that conditional probability may be defined even when it is conditional on a 0-probability event.

equation (1) in a way similarly linked to the truth conditions of $\mathbf{P}(q \mid p)$ and $\mathbf{P}(\neg q \mid p)$? De Finetti theory of tri-events aims at providing a positive answer to questions like this. As we shall see, the original theory proposed by de Finetti does not allow this conclusion, while the modified modal theory we propose does.

Another route to tri-events comes from consideration of bets. If a bet is on a two-valued event E, the bettor wins the bet if E obtains, that is, if the sentence that expresses it is true and is lost if it is false. This remark suggests that bets are at the intersection of sentence logic and probability. De Finetti models conditional probabilities on an event E given another event H by conditional bets, that is, bets that the bettor wins if both H and E occur, that the bettor loses if H occurs, but E does not, and that is called for if H does not occur. De Finetti developed his theory of probability in terms of coherence in the betting behaviour. In such a framework, he identifies the probability of an event E with that betting quotient which the bettor considers 'fair', that is at which the bettor is indifferent between betting on E or betting against E.[6] As far as conditional bets are concerned, if winning is associated with truth and losing with falsehood, calling for the bet is associated with a truth-value gap. In this way, a bet on an event E conditional to another event H is seen as a bet on a *conditional event* $(E \mid H)$ which is expressed by a conditional sentence, meant in a suppositional way: "E if H" or "E on the supposition that H". This conditional event expresses a true sentence if both H and E occur, is false if H is true while E is false, and is 'null' if H does not occur. So the truth-conditional semantics of tri-events is seen as the semantical counterpart of the logic of probability, including conditional probability, in terms of bets.

There is also a third Definettian route to tri-events. According to de Finetti, the probability of a proposition is the expectation of its truth-value (he used the bold letter '**P**' to designate both probability and expectation, renamed 'prevision'). How may we apply this view to conditional probability? The answer resorts to the idea of *conditional expectation*. The conditional expectation is a well-known notion of probability theory. Where X is a random number and p an event, it holds that $\mathbf{P}_p(X) = \frac{\mathbf{P}(X \times p)}{\mathbf{P}(p)}$ if $\mathbf{P}(p) > 0$.

When X is, in turn, an event (say q), we have $\mathbf{P}_p(q) = \frac{\mathbf{P}(q \times p)}{\mathbf{P}(p)} = \mathbf{P}(q \mid p)$.

Although, as we have already said, de Finetti had not the intent of solving a problem about conditional sentences, the theory of tri-events actually provides an *ante litteram* solution to a problem about conditionals with which more recently philosophers have struggled: to find a connective '$\Rightarrow$' such that $\mathbf{P}(p \Rightarrow q) = \mathbf{P}(q \mid p)$. Indeed, if '|' is a connective and $\mathbf{P}(q \mid p)$ equals the conditional probability of q given p, the '|' is just the wanted connective.

As already pointed out, this solution may seem at odds with LRT. However, this is not the case. For, LRT, even in the strongest form proved by Hájek (1989), assumes that conditionals are two-valued propositions, a premise that in the de Finetti's theory does not hold.

[6] If, as de Finetti does, one assumes that the final goal of the bettor is maximising expected gain (supposed linear in utility), one may equivalently characterise the fairness of bets as indifference between betting on E and abstaining from betting. The characterisation in the text does not require that rule of maximisation of expected gain is satisfied (see Mura 1994).

The standard axioms of probability apply to Boolean algebras. Defining probability on other structures requires some adjustment (as happens, for example, in Quantum Mechanics). Since, algebraically, the field of tri-events is a lattice but not, in general, a Boolean algebra, it is quite natural that the laws of probability must be adjusted accordingly. Such a lattice contains as a sub-lattice the Boolean algebra of two-valued events. Of course, as far as two-valued ordinary events are concerned, the usual laws of probability must be satisfied. Since ordinary events are special cases of tri-events, there is no harm in generalised probability laws that turn out to coincide with the usual laws when only ordinary events are involved. In section 6.1 we provide a new set of probability axioms that apply in a general way to tri-events and are such that reduce to the standard axioms of conditional probability when only ordinary events are involved.

As remarked above, de Finetti's presents his theory as a three-valued logic, where the third value beyond true and false is meant as a truth-value gap. Truth-tables define the semantics of connectives. De Finetti takes negation, disjunction and conjunction from Łukasiewicz (1930) 1987 three-valued logic. The implication is the same as in Kleene 1938 strong three-valued logic (though de Finetti introduced it before Kleene). The qualifying connective is conditioning, whose truth-table I have reported above. Along with these three-valued connectives, de Finetti's semantics contains, like in Bochvar (1937) 1981, two unary connectives that transform a three-valued event in a two-valued one. We define them by the following truth-table:[7]

	Thesis	Hypothesis
p	$\uparrow p$	$\updownarrow p$
1	1	1
0	0	1
u	0	0

These connectives are important in many respects.[8] First, they allow to show that every sentence that expresses a compound of tri-events has the same truth conditions of a sentence that expresses a *simple tri-event*, that is a sentence of the form $(\varphi \mid \psi)$ where both φ and ψ are two-valued sentences. In fact, in general it holds the equivalence $\chi = (\uparrow \chi \mid \updownarrow \chi)$. Second, they allow the algebra of tri-events to contain the Boolean algebra of binary events (whose elements are those tri-events φ such that $\varphi = \uparrow \varphi$). As far as probability is concerned, Bayes Theorem in the standard form $\mathbf{P}(\psi \mid \varphi) = \frac{\mathbf{P}(\psi \wedge \varphi)}{\mathbf{P}(\varphi)}$ (provided $\mathbf{P}(\varphi) > 0$) does not hold in general for tri-events, albeit it holds with two-valued tri-event. However, the more general formula $\mathbf{P}(\varphi) = \frac{\mathbf{P}(\uparrow \varphi)}{\mathbf{P}(\updownarrow \varphi)}$ (provided $\mathbf{P}(\updownarrow \varphi) > 0$) is always satisfied.

[7] The symbolism used here differs from de Finetti's symbolism. He used 'T' (for 'Thesis') instead of our '$\uparrow$' and 'H' (for 'Hypothesis') instead of our '$\updownarrow$'.

[8] In what follows I will call *events* or *propositions* the equivalence classes defined on the set of sentences induced by the relation of having the same truth conditions.

Is de Finetti's semantics compatible with Adams' probabilistic semantics? Can we define a logical consequence relation coextensive with Adams p-entailment? The answer is in the negative. This result follows from a theorem by Mc Gee (1981), according to which no m-valued truth-functional logic, even if equipped with modal operators, allows the definition of a relation of logical consequence preserving designated values and coextensive to p-entailment. As already pointed out, in the case of de Finetti's semantics, this is connected to the fact that tri-events of the form $(\varphi \mid \varphi)$ are *not* valid formulas. Tri-events of the form $(\varphi \mid \varphi)$ are quasi-tautologies, that is they may be either true or null. We cannot convene to equate quasi-tautologies to tautologies in a general way because those tri-events that are necessarily null (call it *singular tri-events*, see below Definition 2.6) are quasi-tautologies and are also truth-functionally equivalent to their negation, so that, in such a case, the negation of a tautology would be a tautology, which would amount to an inconsistency.

2. The Modal Semantics of Tri-events: Preliminary Explanations

The modal semantics proposed here aims at defining truth-conditional semantics for tri-events. Such semantics includes a logical consequence relation that generalises the common logical consequence for two-valued ordinary sentences and is coextensive with Adams p–entailment (extended to the lattice of tri-events). According to this semantics, a sentence of the form $(\varphi \mid \varphi)$ is not true only when φ is either singular or logically false. As anticipated above, we may express this logical fact by the schema $(\Diamond \updownarrow \varphi \;\rightarrow \uparrow (\varphi \mid \varphi))$ (where '$\rightarrow$' is a special material conditional, whose truth conditions will be specified below (see page 304).

The basic idea consists of defining the truth conditions so that no sentence is not false at every world and true at some but not at all worlds. In the same vein, no sentence fails to be true at every world and false at some world that is not false at every world. Underlying this tenet is the idea that, in the context of partial logic, a sentence φ must be considered as valid (unsatisfiable) if and only the two following conditions are satisfied: (a) φ is true (false) at all possible worlds in which it has a truth value and (b) there is a possible world at which φ is true (false). Clause (a) is required if we take seriously the idea that a sentence that is neither true nor false lacks a truth-value rather than bearing a third genuine truth-value. Clause (b) is required because sentences that cannot be true (false) would be considered as necessarily true (false), which seems absurd. This notion of validity (unsatisfiability) reduces to the standard notion when φ is a two-valued sentence. The truth conditions of atomic sentences are re-valuated in such a way that those atomic sentences that satisfy clauses (a) and (b) are considered as true (false) at every world. [9] Moreover, the truth conditions of some connectives are warped in such a way that molecular sentences satisfying clauses (a) and (b) turn out to be

[9] This re-valuation resembles van Fraassen's *supervaluation* (1966). However, it differs in several respects and affects the truth conditions of molecular sentences in a general way. Elsewhere (Mura 2009) I have called my re-evaluation of sentences 'hypervaluations.' In the present Kripke-style framework, however, no formal definition of hypervaluations is required.

true (false) at every world. Connectives so modified have a modal import, and modal operators are definable by them.

So, by this semantics, sentences are exhaustively so classified concerning a given model:

(a) Necessarily true (true at every world);
(b) Necessarily false (false at every world);
(c) Singular (neither true nor false at every word);
(d) Contingent (true at some world and false at some world).

Among the contingent sentences another distinction is relevant: the distinction between those contingent sentences that at every world are either true or false (we call *two-valued* such sentences) and those contingent sentences that lack a truth-value at one or more worlds. The class of two-valued contingent sentences joined with the class of necessarily true sentences and with the class of necessarily false sentences is closed under all connectives except '|' conditioning connective. So the present semantics extends the standard sentential modal S5 semantics rather than replacing it. Atomic sentences may belong to each of the categories above. If they are contingent, they can be two-valued or not, depending on the model.

A special feature of this semantics is that *some of the binary connectives carry a modal import*, i.e. their truth conditions refer to the set of possible worlds. As a result, we may define modal operators in terms of the other connectives. However, we prefer to put explicitly the truth conditions of modal operators. By contrast, all unary connectives are truth-functional. We stress that the modal aspects of binary connectives do not affect completely independent sets of non-singular sentences[10] so that in many examples discussed in the literature the truth conditions of these connectives coincide with the original de Finetti's truth conditions. The main modification of the proposed semantics consists in considering true (false) at a given world a sentence which, according to de Finetti's semantics, would be null whenever it is not false (true) at every world and true (false) at some world.

2.1 The Modal Semantics in Detail

2.1.1 The Language $\mathscr{L}$

Constants: $\bot$, $\neg$. $\uparrow$, $\vee$, $\wedge$, $|$, $\rightarrow$, $\diamond$
Atomic sentences: $\mathbb{P}_0$, $\mathbb{P}_1$,...

Definition 2.1 (Sentences)

The string φ of symbols of $\mathscr{L}$ is a sentence iff at least one of the following conditions is satisfied (where the metalinguistic symbol '$\approx$' means 'has the same form as'):

(a) φ is an atomic sentence;

[10] A set of non-singular sentences U is *completely independent* iff for arbitrary disjoint finite subsets $\{u_1,\dots,u_n\}$, and $\{v_1,\dots,v_m\}$ of U, the set $\{u_1,\dots,u_n, \neg v_1,\dots,\neg v_m\}$ is satisfiable (see definitions 2.7 and 2.8 below).

(b) φ = '$\bot$';

(c) either $\varphi \approx \neg\psi$ or $\varphi \approx\uparrow \psi$ or $\varphi \approx \Diamond\psi$, where ψ is a sentence;

(d) either $\varphi \approx (\psi \vee \chi)$ or $(\psi \wedge \chi)$ or $\varphi \approx (\psi \rightarrow \chi)$ where both ψ and χ are sentences;

(e) $\varphi \approx (\psi \mid \chi)$ where both ψ and χ are sentences.

Definition 2.2 (Metalinguistically defined logical constants)

$$\top \quad \stackrel{\text{def}}{=} \quad (\bot \rightarrow \bot)$$

$$\natural \quad \stackrel{\text{def}}{=} \quad (\bot \mid \bot)$$

$$(\varphi \leftrightarrow \psi) \quad \stackrel{\text{def}}{=} \quad ((\varphi \rightarrow \psi) \wedge (\psi \rightarrow \varphi))$$

$$\downarrow \varphi \quad \stackrel{\text{def}}{=} \quad (\neg\uparrow\varphi \wedge \neg\uparrow\neg\varphi)$$

$$\updownarrow \varphi \quad \stackrel{\text{def}}{=} \quad (\uparrow\varphi \vee \uparrow\neg\varphi)$$

$$\Box\varphi \quad \stackrel{\text{def}}{=} \quad \neg\Diamond\neg\varphi$$

The set of the primitive logical constants of $\mathscr{L}$ is redundant. For example, according to the semantic definition below, $\Diamond$ is definable, since the following equivalence holds at every world in every model: $\Diamond\varphi \leftrightarrow ((\top \rightarrow (\varphi \mid \varphi)) \mid \uparrow(\varphi \vee \neg\varphi))$. However, in a semantical context, this redundancy presents several well-known advantages. The following definition is adapted from Chellas 1980. We symbolise that a sentence φ is true (false) at world α in the model $\mathcal{M}$ by $\models^{\mathcal{M}}_{\alpha} \varphi$ ($\overline{\models}^{\mathcal{M}}_{\alpha} \varphi$).

Definition 2.3 (Models)

A model is a triplet $\mathcal{M} = (W, P, Q)$, where W is a set, whose elements are called 'worlds', P and Q are functions of natural numbers such that for each number n, P_n and Q_n are subsets of W (i.e. $P : \mathbb{N} \rightarrow \wp(W)$; $Q : \mathbb{N} \rightarrow \wp(W)$), and for each n, Q_n is a subset of P_n.

Definition 2.4 (Truth-conditions at a possible world α in the model $\mathcal{M}$)

Let α be a world in a model $\mathcal{M} = (W, P, Q)$.

1.

(a) $\mathbb{P}_n$ is true at α (where $n \in \mathbb{N}$): $\models^{\mathcal{M}}_{\alpha} \mathbb{P}_n$ iff either $\alpha \in Q_n$ or both the following conditions are satisfied:

(i) $P_n = Q_n$;

(ii) $Q_n \neq \emptyset$.

(b) $\mathbb{P}_n$ is false at α: $\overline{\models}^{\mathcal{M}}_{\alpha} \mathbb{P}_n$ iff either $\alpha \in P_n - Q_n$ or both the following conditions are satisfied:

(i) $Q_n = \emptyset$
(ii) $P_n \neq \emptyset$.

COMMENT. Atomic sentences may be, at any world, either true or false or neither true nor false. Every atomic sentence $\mathbb{P}_n$ has associated two subsets of the set of worlds, the set P_n and the set Q_n. P_n is *a* set of worlds in which the atomic sentence $\mathbb{P}_n$ is either true or false, Q_n is *a* set of worlds in which the atomic sentence $\mathbb{P}_n$ is true (where $Q_n \subseteq P_n$). If P_n and Q_n would contain as elements, respectively, *all* the true atomic sentences and *all* those atomic sentences that are either true or false, then they, taken together, would define a common valuation V of atomic sentences. In our semantics, however, sentences that do not belong to Q_n (so that by V they are neither true nor false) are re-valuated in such a way that even if $\alpha \notin P_n$, $\mathbb{P}_n$ may be true or false. In such a case, $\mathbb{P}_n$ is true at α if $\mathbb{P}_n$ is true at some world $\in Q_n$ and false at no world $\in P_n$, and $\mathbb{P}_n$ is false at α if $\mathbb{P}_n$ is false at some world $\in P_n$ and true at no world $\in Q_n$. The truth conditions of atomic sentences are well-defined: at every world, a sentence is either true or false or neither true nor false. It should be stressed that the truth conditions of atomic sentences at a given world have a modal import since they may depend on conditions about the totality of worlds. The present approach differs from the common treatment of conditional events that typically moves from atomic two-valued sentences (or events). The two approaches are, after all, equivalent and differ only in technical details. However, considering atomic sentences that may lack a truth-value is in a better agreement with the view that two-valued sentences are just a special case and not the basic sentences upon which tri-events are built.

2.
(a) Not $\models^{\mathcal{M}}_{\alpha} \bot$;
(b) $\overline{\models}^{\mathcal{M}}_{\alpha} \bot$.

COMMENT. '$\bot$' is false at every world.

3.
(a) $\models^{\mathcal{M}}_{\alpha} \neg\varphi$ iff $\overline{\models}^{\mathcal{M}}_{\alpha} \varphi$;
(b) $\overline{\models}^{\mathcal{M}}_{\alpha} \neg\varphi$ iff $\models^{\mathcal{M}}_{\alpha} \varphi$.

COMMENT. $\neg\varphi$ is true at those worlds at which φ is false and false at those worlds at which φ is true. The semantics of '$\neg$' carries no modal import beyond the possible modal import of the involved atomic sentences, being truth-functional.

4.
(a) $\models^{\mathcal{M}}_{\alpha} \uparrow\varphi$ iff $\models^{\mathcal{M}}_{\alpha} \varphi$;
(b) $\overline{\models}^{\mathcal{M}}_{\alpha} \uparrow\varphi$ iff not $\models^{\mathcal{M}}_{\alpha} \varphi$.

COMMENT. $\uparrow\varphi$ is true at a world α iff φ is true at α and false at α otherwise. So '$\uparrow\varphi$' is a two-valued sentence. The semantics of '$\uparrow$' carries no modal import, being truth-functional.

5.
  (a) $\models^{\mathcal{M}}_{\alpha} (\varphi \vee \psi)$ iff either $\models^{\mathcal{M}}_{\alpha} \varphi$ or $\models^{\mathcal{M}}_{\alpha} \psi$ or all the following conditions are satisfied:
    (i) for no β in $\mathcal{M}$ it holds that $\overline{\models}^{\mathcal{M}}_{\beta} \varphi$ and $\overline{\models}^{\mathcal{M}}_{\beta} \psi$;
    (ii) there exists β in $\mathcal{M}$ such that either $\models^{\mathcal{M}}_{\beta} \varphi$ or $\models^{\mathcal{M}}_{\beta} \psi$.
  (b) $\overline{\models}^{\mathcal{M}}_{\alpha} (\varphi \vee \psi)$ iff both $\overline{\models}^{\mathcal{M}}_{\alpha} \varphi$ and $\overline{\models}^{\mathcal{M}}_{\alpha} \psi$ or all the following conditions are satisfied:
    (i) for no β in $\mathcal{M}$ it holds that $\models^{\mathcal{M}}_{\beta} \varphi$ or $\models^{\mathcal{M}}_{\beta} \psi$;
    (ii) there exists β in $\mathcal{M}$ such that both $\overline{\models}^{\mathcal{M}}_{\beta} \varphi$ and $\overline{\models}^{\mathcal{M}}_{\alpha} \psi$.

COMMENT. $(\varphi \vee \psi)$ is a modified version of Łukasiewicz three-valued disjunction (where the third truth-value out of true and false is interpreted as a truth-value gap). More exactly, in a model $\mathcal{M}$, $(\varphi \vee \psi)$ is true at world α if either (a) φ or ψ is true at α or (b) both φ and ψ are not false at every world and either φ or ψ is true at some world. $(\varphi \vee \psi)$ is false at α if either (a) neither φ nor ψ are true at α or (b) $(\varphi \vee \psi)$ is false at α and there is no world at which either φ or ψ is true and there is a world at which both φ and ψ are false. Notice that the disjunction connective typically has a modal import, since its truth conditions may depend not only on the truth-values of the involved sentences but also on the truth-valued across the totality of worlds.

6.
  (a) $\models^{\mathcal{M}}_{\alpha} (\varphi \wedge \psi)$ iff either both $\models^{\mathcal{M}}_{\alpha} \varphi$ and $\models^{\mathcal{M}}_{\alpha} \psi$ or both the following conditions are satisfied:
    (i) For no world β in $\mathcal{M}$ it holds that either $\overline{\models}^{\mathcal{M}}_{\alpha} \varphi$ or $\overline{\models}^{\mathcal{M}}_{\alpha} \psi$
    (ii) There is a world β in $\mathcal{M}$ such that $\models^{\mathcal{M}}_{\alpha} \varphi$ and $\models^{\mathcal{M}}_{\alpha} \psi$
  (b) $\overline{\models}^{\mathcal{M}}_{\alpha} (\varphi \wedge \psi)$ iff either $\overline{\models}^{\mathcal{M}}_{\alpha} \varphi$ or $\overline{\models}^{\mathcal{M}}_{\alpha} \psi$ or all the following conditions are satisfied:
    (i) For no β in $\mathcal{M}$ it holds that $\models^{\mathcal{M}}_{\beta} \varphi$ and $\models^{\mathcal{M}}_{\beta} \psi$;
    (ii) at least one of the following conditions is satisfied:
      (A) there exists γ in $\mathcal{M}$ such that $\overline{\models}^{\mathcal{M}}_{\gamma} \varphi$
      (B) there exists γ in $\mathcal{M}$ such that $\overline{\models}^{\mathcal{M}}_{\gamma} \psi$.

COMMENT. $(\varphi \wedge \psi)$ is a modified version of Łukasiewicz three-valued conjunction (where the third truth-value out of true and false is interpreted as a truth-value gap). More exactly, in a model $\mathcal{M}$, $(\varphi \wedge \psi)$ is true at a world α if both φ and ψ are true at α. $(\varphi \wedge \psi)$ is also true at α if at no world either φ or ψ are false, and there is a world at which both φ and ψ are true. $(\varphi \wedge \psi)$ is false at α if either φ or ψ is false at α or φ, and ψ are not both true at any world and either there is a world at which φ is false, or there is a world at which ψ is false. From theorem 13 below it follows that $\models^{\mathcal{M}}_{\alpha} (\varphi \wedge \psi)$ iff $\models^{\mathcal{M}}_{\alpha} \varphi$ and $\models^{\mathcal{M}}_{\alpha} \psi$. Notice that the conjunction connective, like disjunction, has a modal import since its truth conditions depend not only on the truth-values of the involved sentences but also on the truth-valued across the totality of worlds.

7.
(a) $\models^{\mathcal{M}}_{\alpha} (\varphi \mid \psi)$ iff either $\models^{\mathcal{M}}_{\alpha} \varphi$ and $\models^{\mathcal{M}}_{\alpha} \psi$ or the two following conditions are satisfied:

   (i) there exists β in $\mathcal{M}$ such that $\models^{\mathcal{M}}_{\beta} \varphi$ and $\models^{\mathcal{M}}_{\beta} \psi$;

   (ii) for no β in $\mathcal{M}$ it holds that $\models^{\mathcal{M}}_{\beta} \psi$ and $\overline{\models}^{\mathcal{M}}_{\beta} \varphi$.

(b) $\overline{\models}^{\mathcal{M}}_{\alpha} (\varphi \mid \psi)$ iff either both $\models^{\mathcal{M}}_{\alpha} \psi$ and $\overline{\models}^{\mathcal{M}}_{\alpha} \varphi$, or the two following conditions are satisfied:

   (A) there exists β in $\mathcal{M}$ such that $\models^{\mathcal{M}}_{\beta} \psi$ and $\overline{\models}^{\mathcal{M}}_{\beta} \varphi$;

   (B) for no β in $\mathcal{M}$ it holds that $\models^{\mathcal{M}}_{\beta} \varphi$ and $\models^{\mathcal{M}}_{\beta} \psi$.

COMMENT. $(\varphi \mid \psi)$ is a modified version of de Finetti's conditioning. In a model $\mathcal{M}$, $(\varphi \mid \psi)$ is true at α if both φ and ψ are true at α. Moreover, $(\varphi \mid \psi)$ is true at α if at no world β, ψ is true at β while φ is false at β *and* there exists a world at which both φ and ψ are true. $(\varphi \mid \psi)$ is false at α if ψ is true at α while φ is false at α. Moreover, $(\varphi \mid \psi)$ is false at α if there is no world β such that ψ is true at β and φ is false at β *and* there is no world at which both φ and ψ are true. Notice that also the conditioning connective has a modal import, since its truth conditions depend not only on the truth-values of the involved sentences, but also on the truth-valued across the totality of worlds.

8.
(a) $\models^{\mathcal{M}}_{\alpha} (\varphi \to \psi)$ iff at least one of the following conditions are satisfied:

   (i) $\overline{\models}^{\mathcal{M}}_{\alpha} \varphi$;
   (ii) $\models^{\mathcal{M}}_{\alpha} \psi$;
   (iii) not $\models^{\mathcal{M}}_{\alpha} \varphi$ and not $\overline{\models}^{\mathcal{M}}_{\alpha} \psi$.

(b) $\overline{\models}^{\mathcal{M}}_{\alpha} (\varphi \to \psi)$ iff at least one of the following conditions are satisfied:

   (i) $\models^{\mathcal{M}}_{\alpha} \varphi$ and not $\models^{\mathcal{M}}_{\alpha} \psi$;
   (ii) not $\overline{\models}^{\mathcal{M}}_{\alpha} \varphi$ and $\overline{\models}^{\mathcal{M}}_{\alpha} \psi$.

COMMENT. $(\varphi \to \psi)$ is a special material conditional. $(\varphi \to \psi)$, at given world α in a model $\mathcal{M}$, is either true or false, so it is a two-valued sentence. $(\varphi \to \psi)$ is true at α if the antecedent φ is false at α or the consequent ψ is true at α. In addition, $(\varphi \to \psi)$ is true at α if φ is not true at α and ψ is not false at α. On the other hand, $(\varphi \to \psi)$ is false at α if φ is true at α and ψ is not true at α. Moreover, $(\varphi \to \psi)$ is false at α if φ is not false at α while ψ is false at α. The semantics of '$\to$' carries *no* modal import, being truth-functional.

9.
(a) $\models^{\mathcal{M}}_{\alpha} \Diamond\varphi$ iff there exists β in $\mathcal{M}$ such that $\models^{\mathcal{M}}_{\alpha} \varphi$;
(b) $\overline{\models}^{\mathcal{M}}_{\alpha} \Diamond\varphi$ iff for no β in $\mathcal{M}$ it holds that $\models^{\mathcal{M}}_{\beta} \varphi$ and there exists β in $\mathcal{M}$ such that $\overline{\models}^{\mathcal{M}}_{\alpha} \varphi$.

COMMENT. $\Diamond\varphi$ is true iff φ is true at some world. $\Diamond\varphi$ is false at α iff it is true at no world and false at some world. So, $\Diamond\varphi$ is neither true nor false only if φ is neither true nor false at every world. Note that $\Diamond\varphi$ does not say that φ is possibly true but conditionally says that φ is possibly true supposing that it is not the case that φ is singular (that is neither true nor false at every world). Indeed, if φ is singular (see 2.6 below), neither condition (a) nor condition (b) is satisfied at any world, so that $\Diamond\varphi$ is neither true nor false at every world.

Theorem 2.1 (Truth-conditions for defined connectives)

Let α be a world in a model $\mathcal{M} = (W, P, Q)$. The truth conditions for non–primitive logical constants are the following (proofs omitted):

1
- (a) $\models_\alpha^{\mathcal{M}} \top$;
- (b) Not $\overline{\models}_\alpha^{\mathcal{M}} \top$.

COMMENT. $\top$ is a constant that is true at every world.

2
- (a) Not $\models_\alpha^{\mathcal{M}} \natural$;
- (b) Not $\overline{\models}_\alpha^{\mathcal{M}} \natural$.

COMMENT. $\natural$ is a constant that is neither true nor false at every world.

3
- (a) $\models_\alpha^{\mathcal{M}} (\varphi \leftrightarrow \psi)$ iff at least one of the following conditions are satisfied:
  - (i) $\models_\alpha^{\mathcal{M}} \varphi$ and $\models_\alpha^{\mathcal{M}} \psi$;
  - (ii) Not $\models_\alpha^{\mathcal{M}} \varphi$, not $\overline{\models}_\alpha^{\mathcal{M}} \varphi$, not $\models_\alpha^{\mathcal{M}} \psi$, and not $\overline{\models}_\alpha^{\mathcal{M}} \psi$;
  - (iii) $\overline{\models}_\alpha^{\mathcal{M}} \varphi$ and $\overline{\models}_\alpha^{\mathcal{M}} \psi$.
- (b) $\overline{\models}_\alpha^{\mathcal{M}} (\varphi \leftrightarrow \psi)$ iff one of the following conditions are satisfied:
  - (i) $\models_\alpha^{\mathcal{M}} \varphi$ and not $\models_\alpha^{\mathcal{M}} \psi$;
  - (ii) $\models_\alpha^{\mathcal{M}} \psi$ and not $\models_\alpha^{\mathcal{M}} \varphi$
  - (iii) not $\models_\alpha^{\mathcal{M}} \varphi$, not $\overline{\models}_\alpha^{\mathcal{M}} \varphi$, and $\models_\alpha^{\mathcal{M}} \psi$;
  - (iv) not $\models_\alpha^{\mathcal{M}} \varphi$, not $\overline{\models}_\alpha^{\mathcal{M}} \varphi$, and $\overline{\models}_\alpha^{\mathcal{M}} \psi$;
  - (v) $\overline{\models}_\alpha^{\mathcal{M}} \varphi$ and not $\overline{\models}_\alpha^{\mathcal{M}} \psi$;
  - (vi) not $\overline{\models}_\alpha^{\mathcal{M}} \varphi$ and $\overline{\models}_\alpha^{\mathcal{M}} \psi$.

COMMENT. $(\varphi \leftrightarrow \psi)$ is two-valued. In a given model $\mathcal{M}$, $(\varphi \leftrightarrow \psi)$ is true at the world α iff φ and ψ are either both true or both false at α or both neither true nor false at α. It is false at α otherwise. In other terms, $(\varphi \leftrightarrow \psi)$ is true if φ and ψ have the same truth conditions in $\mathcal{M}$ and it is false otherwise.

4
- (a) $\models_\alpha^{\mathcal{M}} \downarrow \varphi$ iff not $\models_\alpha^{\mathcal{M}} \varphi$ and not $\overline{\models}_\alpha^{\mathcal{M}} \varphi$;
- (b) $\overline{\models}_\alpha^{\mathcal{M}} \downarrow \varphi$ iff $\models_\alpha^{\mathcal{M}} \varphi$ or $\overline{\models}_\alpha^{\mathcal{M}} \varphi$.

COMMENT. $\downarrow\varphi$ is true at given world α iff φ is neither true nor false at α and it is false iff φ is either true or false at α. So, $\downarrow\varphi$ is a two-valued sentence. The semantics of '$\downarrow$' carries no modal import, being truth-functional.

5
(a) $\models^{\mathcal{M}}_{\alpha} \updownarrow \varphi$ iff $\models^{\mathcal{M}}_{\alpha} \varphi$ or $\overline{\models}^{\mathcal{M}}_{\alpha} \varphi$;
(b) $\overline{\models}^{\mathcal{M}}_{\alpha} \updownarrow \varphi$ iff not $\models^{\mathcal{M}}_{\alpha} \varphi$ and not $\overline{\models}^{\mathcal{M}}_{\alpha} \varphi$.

COMMENT. In a given model $\mathcal{M}$, $\updownarrow\varphi$ is true at given world α iff φ is either true or false at α and it is false at α iff φ is neither true nor false at α. So, $\updownarrow\varphi$ is a two-valued sentence. The semantics of '$\updownarrow$' carries no modal import, being truth-functional.

6
(a) $\models^{\mathcal{M}}_{\alpha} \Box\varphi$ iff for no β in $\mathcal{M}$ it holds that $\overline{\models}^{\mathcal{M}}_{\beta} \varphi$ and there exists β in $\mathcal{M}$ such that $\models^{\mathcal{M}}_{\beta} \varphi$;
(b) $\overline{\models}^{\mathcal{M}}_{\alpha} \Box\varphi$ iff there is β in $\mathcal{M}$ such that $\overline{\models}^{\mathcal{M}}_{\beta} \varphi$.

COMMENT. In a given model $\mathcal{M}$, $\Box\varphi$ is true iff φ is false at no world and there is word in $\mathcal{M}$ at which φ is true. $\Box\varphi$ is false if φ is false at some world. $\Box\varphi$ is neither true nor false only if φ is neither true nor false at every world. Note that $\Box\varphi$ does not *categorically* says that φ is necessarily true, but it *conditionally* says that φ is necessarily true under the condition that it is not the case that φ is neither true nor false at every world. Indeed, if φ is singular (see 2.6 below), neither condition (a) nor condition (b) is satisfied at any world, so that $\Box\varphi$ is neither true nor false at every world. However, one may express that φ is necessarily true by the formula $\Box\uparrow\varphi$.

Definition 2.5 (Singular sets of sentences in a model $\mathcal{M}$)

A finite set of sentences Γ is said to be singular in $\mathcal{M}$ iff for every world α in $\mathcal{M}$ and every sentence φ of Γ it holds that not $\models^{\mathcal{M}}_{\alpha} \varphi$ and not $\overline{\models}^{\mathcal{M}}_{\alpha} \varphi$.

Definition 2.6 (Singular sentences in a model $\mathcal{M}$)

A sentence φ is said to be singular in a model $\mathcal{M}$ iff the set $\{\varphi\}$ is singular.

Definition 2.7 (Satisfiability in a model $\mathcal{M}$)

A set Γ of sentences is satisfiable in the model $\mathcal{M}$ iff either (i) Γ is empty or (ii) Γ is singular in $\mathcal{M}$, or (iii) for every nonempty subset Γ' of Γ there is a world α in $\mathcal{M}$ such that for every element φ in Γ' it holds that not $\overline{\models}^{\mathcal{M}}_{\alpha} \varphi$ and there is an element ψ in Γ' such that $\models^{\mathcal{M}}_{\alpha} \psi$.

COMMENT. Singular sets are included among satisfiable sets by pure convention. Unless using a partial logic even at the metalinguistic level, we have to decide whether singular sets are satisfiable or unsatisfiable. Our choice presents some technical advantage, but it is otherwise harmless.

Definition 2.8 (Logical satisfiability)

A set Γ of sentences is logically satisfiable iff it is satisfiable in every model $\mathcal{M}$.

Definition 2.9 (Unsatisfiability in a model $\mathcal{M}$)

A set Γ of sentences is unsatisfiable in the model $\mathcal{M}$ iff Γ is not satisfiable in $\mathcal{M}$.

Definition 2.10 (Logical unsatisfiability)

A set Γ of sentences is logically unsatisfiable iff it is unsatisfiable in every model.

Theorem 2.2 (Compactness fails)

There are a model $\mathcal{M}$ and an infinite set A of sentences such that every finite subset of A is satisfiable while A is not satisfiable.

Proof
Assume that $\mathcal{M}$ contains a denumerable ordered set B of atomic sentences $\mathbb{P}_{i_0}$, $\mathbb{P}_{i_1}$,... such that for every i_j every world belongs to P_{i_j} (so that every element of B is two-valued in $\mathcal{M}$) and for at least a k, $Q_{i_k} \neq \emptyset$. Now, let C be the set of all sentences of the form $(\neg\mathbb{P}_{i_j} \wedge \mathbb{P}_{i_{j+1}}) \mid (\mathbb{P}_{i_j} \vee \mathbb{P}_{i_{j+1}}))$. We prove:

(a) C is not logically satisfiable;

(b) every finite subset of C is satisfiable.

(a) By hypothesis the set of worlds in $\mathcal{M}$ at which at least one element of B is true is not empty. Consider the set V of such worlds. At every world, if $\mathbb{P}_{i_j}$ is true then $(\neg\mathbb{P}_{i_j} \wedge \mathbb{P}_{i_{j+1}}) \mid (\mathbb{P}_{i_j} \vee \mathbb{P}_{i_{j+1}}))$ is false by definition 2.4. Now, at every world $\in V$, there is a k such that $(\neg\mathbb{P}_{i_k} \wedge \mathbb{P}_{i_{k+1}}) \mid (\mathbb{P}_{i_k} \vee \mathbb{P}_{i_{k+1}}))$ is false. This proves that C, by definition 2.10 is not logically satisfiable.

(b) Let A be a finite subset $\mathbb{P}_{i_1},\ldots,\mathbb{P}_{i_r}$ of B ($r \geq 2$). Let $\mathcal{M}$ be such that at some world α in $\mathcal{M} \models^{\mathcal{M}}_{\alpha} \mathbb{P}_{i_r}$ and for every w such that $1 \leq w < r$ it holds that $\overline{\models}^{\mathcal{M}}_{\alpha}\ \mathbb{P}_{i_w}$. By definition 2.4 for every w such that $1 \leq w < r-1$ $(\neg\mathbb{P}_{i_w} \wedge \mathbb{P}_{i_{w+1}}) \mid (\mathbb{P}_{i_w} \vee \mathbb{P}_{i_{w+1}}))$ is neither true nor false at α, while $\models^{\mathcal{M}}_{\alpha}$ $(\neg\mathbb{P}_{i_{r-1}} \wedge \mathbb{P}_{i_r}) \mid (\mathbb{P}_{i_{r-1}} \vee \mathbb{P}_{i_r}))$. Therefore, by definition 2.8 is logically satisfiable.

q.e.d.

COMMENT. The example used in proving theorem 2.2 is adapted from Adams 1975: 51-2. In the light of theorem 2.2 no axiomatic system would be complete if the notion of logical consequence is also defined for infinite sets of premises. In any case, since in this paper, we aim at providing a truth-conditional counterpart of Adams' probabilistic semantics, which is defined only with respect to finite sets of sentences, in what follows we'll consider only finite sets of sentences as Adams does.

Definition 2.11 (Validity in a model $\mathcal{M}$)

φ is valid in the model $\mathcal{M}$ ($\models_{\mathcal{M}}$) iff for every world α in $\mathcal{M}$, $\models^{\mathcal{M}}_{\alpha} \varphi$.

Definition 2.12 (Countervalidity in a model $\mathcal{M}$)

φ is countervalid in the model $\mathcal{M}$ ($\overline{\models}_{\mathcal{M}}$) iff for every world α in $\mathcal{M}$, $\overline{\models}_{\alpha}^{\mathcal{M}}\ \varphi$

Definition 2.13 (Logical validity)

φ is valid ($\models \varphi$) iff for every model $\mathcal{M}$ and every world α in $\mathcal{M}$, $\models_{\alpha}^{\mathcal{M}} \varphi$.

COMMENT. A valid sentence is an instance of a valid schema. In other terms, every sentence obtained by replacing some sub-sentences of a valid sentence with other sentences is again a valid sentence. Hence the qualification 'logical'.

Definition 2.14 (Logical countervalidity)

φ is logically countervalid. For every model $\mathcal{M}$ and every world α in $\mathcal{M}$, $\overline{\models}_{\alpha}^{\mathcal{M}}\ \varphi$.

COMMENT. A logically countervalid sentence is an instance of a countervalid schema. In other terms, every sentence obtained by replacing some sub-sentences of a countervalid sentence with other sentences is again a countervalid sentence.

Definition 2.15 ($\mathcal{M}$–consequence)

φ is an $\mathcal{M}$-consequence of the finite set Γ of sentences ($\Gamma \models_{\mathcal{M}} \varphi$) iff either $\Gamma = \emptyset$ and $\models_{\mathcal{M}} \varphi$ or Γ is unsatisfiable in $\mathcal{M}$ or there exists a subset $\Gamma' = \{\varphi_1, \ldots, \varphi_k\}(k > 0)$ of Γ such that for every world α in $\mathcal{M}$ it holds that: (i) if every $\psi \in \Gamma'$ is not false at α then φ is not false at α, (ii) if every $\psi \in \Gamma'$ is not false at α and at least an element of Γ' is true at α then φ is true at α.

Definition 2.16 ($\mathcal{M}$–equivalence)

The sentences φ and ψ are $\mathcal{M}$–equivalent iff $\{\varphi\} \models_{\mathcal{M}} \psi$ and $\{\psi\} \models_{\mathcal{M}} \varphi$.

Definition 2.17 ($\mathcal{M}$–contingency)

The sentence φ is $\mathcal{M}$–contingent iff there are two worlds α and β in $\mathcal{M}$ such that $\models_{\alpha}^{\mathcal{M}} \varphi$ and $\overline{\models}_{\beta}^{\mathcal{M}}\ \varphi$

Definition 2.18 ($\mathcal{M}$–compatibility)

The sentences φ and ψ are $\mathcal{M}$–compatible iff either both φ and ψ are singular or there is a world α in $\mathcal{M}$ at which $\models_{\alpha}^{\mathcal{M}} \varphi$ and $\models_{\alpha}^{\mathcal{M}} \psi$.

Definition 2.19 (Logical consequence)

φ is a logical consequence of the finite set Γ of sentences ($\Gamma \models \varphi$) iff for every model $\mathcal{M}$, $\Gamma \models_{\mathcal{M}} \varphi$.

Definition 2.20 (Logical equivalence)

The sentences φ and ψ are logically equivalent iff $\{\varphi\} \models \psi$ and $\{\psi\} \models \varphi$.

Theorem 2.3

Two sentences φ and ψ are logically equivalent iff for every model $\mathcal{M}$ and every world α in $\mathcal{M}$ it holds that $\models_\alpha^{\mathcal{M}} (\varphi \leftrightarrow \psi)$.

Proof

Omitted.

q.e.d.

COMMENT. Two logically equivalent sentences have the same truth conditions in every model.

3. Some Fundamental Theorems

Theorem 3.1

If Γ is a finite set of sentences and φ is a nonsingular sentence, then $\Gamma \models_{\mathcal{M}} \varphi$ iff $\Gamma \cup \{\neg\varphi\}$ is unsatisfiable in $\mathcal{M}$.

Proof

Suppose $\Gamma \models_{\mathcal{M}} \varphi$. If Γ is empty then $\models_{\mathcal{M}} \varphi$, so that for every world α it holds that $\overline{\models}_\alpha^{\mathcal{M}} \neg\varphi$. In this case, $\{\varphi\}$ is not satisfiable in $\mathcal{M}$. If Γ is unsatisfiable in $\mathcal{M}$, also $\Gamma \cup \{\neg\varphi\}$ is unsatisfiable in $\mathcal{M}$. If Γ is not empty and it is satisfiable in $\mathcal{M}$, suppose that $\Gamma \cup \{\neg\varphi\}$ is satisfiable in $\mathcal{M}$. For every nonempty subset Γ' of $\Gamma \cup \{\neg\varphi\}$ there is a world α in $\mathcal{M}$ such that for every element ψ in Γ' it holds that not $\overline{\models}_\alpha^{\mathcal{M}} \psi$. Moreover, there is an element χ in Γ' such that $\models_\alpha^{\mathcal{M}} \chi$. Since, by hypothesis, $\Gamma \models_{\mathcal{M}} \varphi$, there is a subset Δ of Γ and a subset S of worlds in $\mathcal{M}$ such that (i) for at least a sentence $\gamma \in \Delta$ and every world $\alpha \in S$ it holds that $\models_\alpha^{\mathcal{M}} \gamma$ and (ii) for no sentence $\gamma \in \Delta$ it holds that $\overline{\models}_\alpha^{\mathcal{M}} \gamma$. Therefore, for every world $\alpha \in S$, it holds that $\models_\alpha^{\mathcal{M}} \varphi$, so that $\overline{\models}_\alpha^{\mathcal{M}} \neg\varphi$ for each $\alpha \in S$. Moreover, at every world $\alpha \notin S$ there is a sentence $\gamma \in \Delta$ such that $\overline{\models}_\alpha^{\mathcal{M}} \gamma$. This contradicts that $\Gamma \cup \{\neg\varphi\}$ is satisfiable in $\mathcal{M}$.

Suppose now that $\Gamma \cup \{\neg\varphi\}$ is unsatisfiable in $\mathcal{M}$. If Γ is unsatisfiable in $\mathcal{M}$ then $\Gamma \models_{\mathcal{M}} \varphi$. So, suppose that Γ is satisfiable. If Γ is empty, then $\{\neg\varphi\}$ is unsatisfiable in $\mathcal{M}$, so that φ is valid in $\mathcal{M}$. Again, it holds that $\Gamma \models_{\mathcal{M}} \varphi$. If Γ is singular and φ is singular too, it holds again that $\Gamma \models_{\mathcal{M}} \varphi$. If Γ is singular and φ is not singular, then $\{\neg\varphi\}$ is unsatisfiable in $\mathcal{M}$, so that $\models_{\mathcal{M}} \varphi$. Again it holds that $\Gamma \models_{\mathcal{M}} \varphi$. Suppose now that Γ is nonempty and nonsingular. By definition, being Γ satisfiable in $\mathcal{M}$, for every nonempty subset Γ' of Γ there is at least one world α in $\mathcal{M}$ such that for every element ψ in Γ' it holds that not $\overline{\models}_\alpha^{\mathcal{M}} \psi$ and there is an element χ in Γ' such that $\models_\alpha^{\mathcal{M}} \chi$. If $\Gamma \cup \{\neg\varphi\}$ is unsatisfiable in $\mathcal{M}$, for every such world α must be $\overline{\models}_\alpha^{\mathcal{M}} \neg\varphi$, so that it holds that $\models_\alpha^{\mathcal{M}} \varphi$ and also $\Gamma \models_{\mathcal{M}} \varphi$.

q.e.d.

Theorem 3.2

Let $\mathcal{M} = (W, P, Q)$ be a model and let φ be a sentence.

(A) If φ is true at some world α, and is false at no world, so that for every β in $\mathcal{M}$ it holds that not $\overline{\models}^{\mathcal{M}}_{\beta}$ φ, then φ is valid in $\mathcal{M}$;

(B) If φ is false at some world α, so that $\overline{\models}^{\mathcal{M}}_{\alpha}$ φ and is true at no world, so that for every β in $\mathcal{M}$ it holds that not $\models^{\mathcal{M}}_{\beta}$ φ, then φ is false at every world in $\mathcal{M}$.

Proof

Let us proceed by induction on the construction of φ.

1. φ is an atomic sentence $\mathbb{P}_n$.

   (A) Since there is a world at which φ is true, it holds that $Q_n \neq \emptyset$. Since φ is false at no world it holds that $P_n = Q_n$. Suppose that φ is not true at any world β. Then $\beta \notin Q_n$. But since $P_n = Q_n$ and $Q_n \neq \emptyset$, by definition 2.4, φ is true at β. So, by *consequentia mirabilis* φ is true at β. So, it is true at all worlds in $\mathcal{M}$. We conclude: $\models_{\mathcal{M}} \varphi$.

   (B) Since there is a world at which φ is false, it holds that $P_n \neq \emptyset$. Since φ is true at no world it holds that $Q_n = \emptyset$. Suppose φ is not false at any world β. So, $\beta \notin P_n$. But since $Q_n = \emptyset$, by definition 2.4, φ is false at β. So, by *consequentia mirabilis* φ is false at β. So, it is false at all worlds. We conclude: $\overline{\models}_{\mathcal{M}} \varphi$.

2. φ = '$\perp$'.

   (A) In this case $\overline{\models}_{\mathcal{M}} \varphi$, so that the condition in the theorem is vacuously satisfied.

   (B) Trivial.

3. $\varphi \approx \uparrow\psi$.

   (A) Since $\uparrow\psi$ is a two-valued sentence, if φ is false at no world, then it is true at every world, so that $\models_{\mathcal{M}} \varphi$.

   (B) Since $\uparrow\psi$ is a two-valued sentence, if φ is true at no world, then it is false at every world, so that $\overline{\models}_{\mathcal{M}} \varphi$.

4. $\varphi \approx \neg\psi$.

   (A) Suppose that φ is true at some world in $\mathcal{M}$, and that φ is false at no world in $\mathcal{M}$. In this case, ψ is false at some world and true at no world in $\mathcal{M}$. By inductive hypothesis ψ is false at all worlds in $\mathcal{M}$. Then φ is true at all worlds in $\mathcal{M}$ and hence $\models_{\mathcal{M}} \varphi$.

   (B) Suppose φ is false at some world in $\mathcal{M}$, and that φ is true at no world in $\mathcal{M}$. In this case, ψ is true at some world in $\mathcal{M}$ and false at no world in $\mathcal{M}$. By inductive hypothesis $\models_{\mathcal{M}} \psi$. Then φ is false at all worlds in $\mathcal{M}$ and hence $\overline{\models}_{\mathcal{M}} \varphi$.

5. $\varphi \approx (\psi \vee \chi)$.

   (A) Suppose φ is true at some world α in $\mathcal{M}$, and that φ is false at no world in $\mathcal{M}$. Suppose that φ is not true at a given world β in $\mathcal{M}$. Neither ψ nor χ are therefore true at β. Now, ψ and χ are not both false at β, otherwise φ would be false at β. So at least one of them (say ψ) is neither true nor false at β. However, either ψ is true at α or χ is true at α or by 2.4 there is a world γ in $\mathcal{M}$ at which either ψ or χ are true. In this last case, by 2.4 ψ and χ are not both false at γ. From this follows by 2.4 that φ is true at β. Contradiction.

   (B) Suppose φ is false at some world α in $\mathcal{M}$, and that φ is true at no world in $\mathcal{M}$. Suppose that φ is not false at a given world β in $\mathcal{M}$. Either ψ or χ are therefore not false at β. Now, either ψ or χ are not true at β, otherwise φ would be true at β. So at least one of them (say ψ) is neither true nor false at β. However, either both ψ and χ are false at α or, in any case, by 2.4 there is a world γ in $\mathcal{M}$ at which both ψ and χ are false. Moreover, there is no world γ in $\mathcal{M}$ at which either ψ or χ are true because in such a case φ would be true at γ. From this follows by 2.4 that φ is false at β. Contradiction.

6. $\varphi \approx (\psi \wedge \chi)$.

   (A) Suppose φ is true at some world α in $\mathcal{M}$, and that φ is false at no world in $\mathcal{M}$. Suppose that φ is not true at a given world β in $\mathcal{M}$. Either ψ or χ (or both) are not true at β. It cannot be $\overline{\models}^{\mathcal{M}}_{\beta}\ \psi$ or $\overline{\models}^{\mathcal{M}}_{\beta}\ \chi$. So, it holds not $\overline{\models}^{\mathcal{M}}_{\beta}\ \psi$ and not $\overline{\models}^{\mathcal{M}}_{\beta}\ \chi$. Then, by definition 2.4 for every world γ in $\mathcal{M}$ it holds that $\models^{\mathcal{M}}_{\gamma} \psi$ and $\models^{\mathcal{M}}_{\gamma} \chi$, so that $\models_{\mathcal{M}} \varphi$.

   (B) Suppose that φ is false at some world α in $\mathcal{M}$, and that φ is true at no world in $\mathcal{M}$. So at some world α either $\overline{\models}^{\mathcal{M}}_{\alpha}\ \varphi$ or $\overline{\models}^{\mathcal{M}}_{\alpha}\ \psi$. Moreover, for every world β it holds that not $\models^{\mathcal{M}}_{\beta} \varphi$ and not $\models^{\mathcal{M}}_{\beta} \psi$. Therefore, by definition 2.4, it holds that $\overline{\models}^{\mathcal{M}}_{\beta}\ \varphi$. So φ is false at every world γ in $\mathcal{M}$ and hence it holds that $\overline{\models}_{\mathcal{M}}\ \varphi$.

7. $\varphi \approx (\psi \mid \chi)$.

   (A) Suppose φ is true at some world α in $\mathcal{M}$, and that φ is false at no world in $\mathcal{M}$. Suppose that φ is not true at a given world β in $\mathcal{M}$. Either ψ or χ (or both) are not true at β. Now, under our assumption is not possible that, at β, χ is true and ψ is false, otherwise φ would be false at β. So, either χ is not true at β or ψ is not false at β. However, either both ψ and χ are true at α or by definition 2.4, there is a world γ in $\mathcal{M}$ such that both χ and ψ are true at γ. By definition 2.4 it is not the case that χ is true at γ and ψ false at γ. From this it follows, by definition 2.4, that φ is true at β. Contradiction.

(B) Suppose φ is false at some world α in $\mathcal{M}$, and that φ is true at no world in $\mathcal{M}$. Suppose that φ is not false at a given world β in $\mathcal{M}$. Therefore, it is not the case that χ is true and ψ false at β. Moreover, it is not the case that both ψ and χ are true at β, otherwise φ would be true at β. However, either χ is true at α and ψ is false at α or, by condition definition 2.4 there is a world γ in $\mathcal{M}$, at which χ is true and ψ is false. Moreover, there is no world γ in $\mathcal{M}$ at which both χ and ψ are true because in such a case φ would be true at γ. From this follows by definition 2.4 that φ is false at β. Contradiction.

8. $\varphi \approx (\psi \rightarrow \chi)$.

(A) Suppose φ is true at some world α in $\mathcal{M}$, and that φ is false at no world in $\mathcal{M}$. Since, at each world in $\mathcal{M}$, φ is either true or false, φ is true at all worlds in $\mathcal{M}$, so that it holds that $\models_{\mathcal{M}} \varphi$.

(B) Suppose φ is false at some world α in $\mathcal{M}$, and that φ is true at no world in $\mathcal{M}$. Since, at each world in $\mathcal{M}$, φ is either true or false, φ is false at all worlds in $\mathcal{M}$, so that it holds that $\overline{\models}_{\mathcal{M}} \varphi$.

9. $\varphi \approx \Diamond\psi$.

(A) Suppose φ is true at some world α in $\mathcal{M}$, and that φ is false at no world in $\mathcal{M}$. Then, by definition 2.4, φ is true at every world α in $\mathcal{M}$, so that it holds that $\models_{\mathcal{M}} \varphi$.

(B) Suppose φ is false at some world α in $\mathcal{M}$, and that φ is true at no world in $\mathcal{M}$. Then, by definition 3, φ is false at every world α in $\mathcal{M}$, so that it holds that $\overline{\models}_{\mathcal{M}} \varphi$.

q.e.d.

COMMENT. In the light of theorem 3.2 quasi-tautologies (that is sentences that at each world α in each model $\mathcal{M}$ are either neither true nor false or true) are singular or valid. They are near-tautologies (see definition 4.2 below.)

Theorem 3.3

Let Γ be a finite set of sentences. Let φ be a sentence. Then $\Gamma \models_{\mathcal{M}} \varphi$ iff $\Gamma \cup \top \models_{\mathcal{M}} \varphi$.

Proof

Suppose that $\Gamma \cup \top \models_{\mathcal{M}} \varphi$. If Γ is empty then $\top \models_{\mathcal{M}} \varphi$, so that φ is $\mathcal{M}$–valid. In this case $\Gamma \models_{\mathcal{M}} \varphi$. Otherwise, let Γ' be a non-empty subset of $\Gamma \cup \top$ such that for every world α in $\mathcal{M}$ it holds that: (i) if every $\psi \in \Gamma'$ is not false at α then φ is not false at α, (ii) if every $\psi \in \Gamma'$ is not false at α and at least an element of Γ' is true at α then φ is true at α. There are two cases: (i) $\top \in \Gamma'$, and (ii) $\top \notin \Gamma'$. In the case (i) φ is $\mathcal{M}$–valid, so that $\Gamma \models_{\mathcal{M}} \varphi$. In the case (ii) it is immediate that $\Gamma \models_{\mathcal{M}} \varphi$.

Suppose $\Gamma \models_{\mathcal{M}} \varphi$. If $\Gamma = \emptyset$ then φ is $\mathcal{M}$–valid, so that $\top \models_{\mathcal{M}} \varphi$ and $\Gamma \cup \top \models_{\mathcal{M}} \varphi$. If $\Gamma \neq \emptyset$, then there is a non-empty subset Γ' of Γ such that (i) if every $\psi \in \Gamma'$ is not false at α then φ is not false at α, (ii) if every $\psi \in \Gamma'$ is not false at α and at least one element of Γ' is true at α, then φ is true at α. Now, Γ' is also a subset of $\Gamma \cup \top$, so that $\Gamma \cup \top \models_{\mathcal{M}} \varphi$.

q.e.d.

4. Quasi S5 Modal System

4.1 Classic Tautologies and Near Tautologies

Definition 4.1 (Classic tautologies)

Let $\mathscr{S}$ be the set of sentences of $\mathscr{L}$ in which the logical constants that occur in them are in the set {'$\perp$', '$\top$', '$\neg$', '$\vee$', '$\wedge$'}. Let $\mathscr{T}$ be the set of sentential tautologies belonging to $\mathscr{S}$. Any element of $\mathscr{S}$ is called a *classic tautology*.

Classic tautological schemas are not, in general, valid schemas in our theory. It is a well-known result that in three-valued logic, tautologies are quasi-tautologies. This result means that instances of tautologous schemas, at every world in every model, are not false, but they may be either true or null at any world. In our theory, we may further sharp this result: every element of $\mathscr{T}$ is either logically valid or logically singular (that is neither true nor false at every world in every model). We call those sentences that have this property *near-tautologies*. Keep in mind that the property of being a near-tautology refers not to a single model but to the set of all models (so that they are instances of near-tautologous schemas).

Definition 4.2 (Near-tautologies)

A sentence φ of $\mathscr{L}$ is called a near-tautology iff for every model $\mathcal{M}$, either $\models_{\mathcal{M}} \varphi$ or φ is singular in $\mathcal{M}$.

Theorem 4.1

Let φ be a classic tautology. Then φ is a near-tautology.

Proof

If φ is a classic tautology then, for every model $\mathcal{M}$ and every world in $\mathcal{M}$ it holds that not $\overline{\models}_{\alpha}^{\mathcal{M}} \varphi$. Now, there are two cases: (a) there is at least a world α in $\mathcal{M}$ at which $\models_{\alpha}^{\mathcal{M}} \varphi$ and (b) for each world α in $\mathcal{M}$ it holds that not $\models_{\alpha}^{\mathcal{M}} \varphi$. In the case (a) by theorem 3.2 $\models_{\mathcal{M}} \varphi$; in the case (b), since it holds that not $\models_{\alpha}^{\mathcal{M}} \varphi$ and not $\overline{\models}_{\alpha}^{\mathcal{M}} \varphi$, φ is a singular sentence. Hence φ is a near-tautology.

q.e.d.

Example. Consider a sentence of the form $(\varphi \vee \neg\varphi)$. If there is a world α in a model $\mathcal{M}$ such that $\models_{\alpha}^{\mathcal{M}} \varphi$ or $\overline{\models}_{\alpha}^{\mathcal{M}} \varphi$ it holds that $\models_{\mathcal{M}} \varphi$. If such a world and model do not exist, then φ is a singular sentence. This is the case when φ is logically equivalent to $\natural$ or $\varphi \approx \chi \mid \psi$, where ψ is logically equivalent to $\bot$.

Near-tautologies do not instance valid schemas. However, they are valid whenever they are not singular sentences. Since a singular sentence logically entails a valid sentence and '$\natural$' is a singular sentence, we have the following result:

Theorem 4.2

Let φ be a near-tautology. Then $\natural \models \varphi$.

Proof

There are two cases. (a) φ is logically valid, and (b) φ is logically singular. In the case (a) for every sentence ψ it holds that $\psi \models \varphi$. In particular, it holds that $\natural \models \varphi$. In the case (b) '$\natural$' and φ have the same truth conditions, so that, $\natural \models \varphi$.

q.e.d.

4.2 Weak Deduction Theorem

Deduction theorem cannot be, in general, satisfied in the present theory. This result is due mainly to the fact that the algebraic structure that underlies our semantics, not being a distributive lattice, is *not* a Heyting algebra. In this case, no material implication satisfying *modus ponens* exists. Despite this, our material implication (that returns a two-valued sentence) satisfies the following weaker property:

Theorem 4.3

For every pair of sentences φ, ψ, and every model $\mathcal{M}$ it holds that $\varphi \models_{\mathcal{M}} \psi$ iff $\models_{\mathcal{M}} (\varphi \to \psi)$.

Proof

Suppose that $\varphi \models_{\mathcal{M}} \psi$. By definition 2.15, at every world α in $\mathcal{M}$, either $\models_\alpha^{\mathcal{M}} \varphi$ and $\models_\alpha^{\mathcal{M}} \psi$ or not $\overline{\models}_\alpha^{\mathcal{M}} \varphi$ and not $\overline{\models}_\alpha^{\mathcal{M}} \psi$ or $\overline{\models}_\alpha^{\mathcal{M}} \varphi$ and $\overline{\models}_\alpha^{\mathcal{M}} \psi$. In each of these case, it holds, by definition 2.4 that $\models_\alpha^{\mathcal{M}} (\varphi \to \psi)$, so that $\models_{\mathcal{M}} (\varphi \to \psi)$.

Suppose that $\models_{\mathcal{M}} (\varphi \to \psi)$. Then at every world α in $\mathcal{M}$ it holds that $\models_\alpha^{\mathcal{M}} (\varphi \to \psi)$. By definition 2.4, either $\models_\alpha^{\mathcal{M}} \varphi$ and $\models_\alpha^{\mathcal{M}} \psi$ or not $\overline{\models}_\alpha^{\mathcal{M}} \varphi$ and not $\overline{\models}_\alpha^{\mathcal{M}} \psi$ or $\overline{\models}_\alpha^{\mathcal{M}} \varphi$ and $\overline{\models}_\alpha^{\mathcal{M}} \psi$. By definition 2.15, it holds that $\varphi \models_{\mathcal{M}} \psi$.

q.e.d.

It should be noted that, in general, *modus ponens* does not hold concerning material conditional $\to$. However, *modus ponens* is always valid if either the antecedent or the consequent is a two-valued sentence (see *infra*). In particular, if φ is valid, from φ and $(\varphi \to \psi)$ follows ψ. So, a deductive system for valid formulas, based on *modus ponens*, is in theory possible.

4.3 Two-Valued Sentences

Along with tri-events that may be at the same time neither true nor false, there are *two-valued* sentences, expressed by sentences that comply with the excluded-middle principle. A sentence may be two-valued in one model $\mathcal{M}$ without being two-valued in all models. Even an atomic sentence $\mathbb{P}_n$ may be two-valued in a model $\mathcal{M} = (W, P, Q)$ if $P_n = W$. A molecular sentence whose atomic sentences are two-valued and whose logical constants occurring in it belong to the set {'$\perp$', '$\vee$', '$\neg$', '$\wedge$','$\rightarrow$', '$\diamond$', '$\uparrow$'} (i.e. the set of all the primitive constants minus {|} is, in turn, two-valued. A lattice of "genuine" tri-events may be generated using conditioning '|' along with the other logical constants from a set of propositions expressed by two-valued sentences. The notion of a two-valued sentence is made precise by the following definition:

Definition 4.3 (Two-valued sentences)

A sentence φ of $\mathscr{L}$ is called a two-valued sentence in a model $\mathcal{M}$ iff for every world α in $\mathcal{M}$ either $\models_\alpha^{\mathcal{M}} \varphi$ or $\overline{\models}_\alpha^{\mathcal{M}} \varphi$

Given a model $\mathcal{M}$, the set of the two-valued sentences of $\mathscr{L}$ in $\mathcal{M}$ is closed concerning the connectives $\neg$, $\vee$, and $\wedge$. Since concerning such sentences, these connectives are Boolean, the set of those propositions that express two-valued sentences of $\mathscr{L}$ in $\mathcal{M}$ form a Boolean algebra.

The notion of a sentence which is two-valued in a given model $\mathcal{M}$, cannot be characterised by syntactic means. Indeed, in a given model $\mathcal{M}$, this property may depend on the truth conditions of atomic sentences in $\mathcal{M}$.

4.4 Essentially Two-Valued Sentences

By the term 'essentially two-valued sentence' we mean a sentence that is two-valued in every model.

Definition 4.4 (Essentially two-valued sentences)

A sentence φ of $\mathscr{L}$ is called an essentially two-valued sentence iff it is a two-valued sentence in every model.

4.5 Ordinary Sentences

Is there any syntactical counterpart of the notion of 'essentially two-valued sentence'? The following results show that the answer is in the positive: there is a recursively syntactically defined class $\mathscr{O}$ of sentences (thereby called *ordinary sentences*), closed under logical constants except $\natural$ and |, whose elements are essentially two-valued. Moreover, for every essentially two-valued sentence, there is an ordinary sentence that is logically equivalent to it.

Definition 4.5 (Ordinary sentences)

φ is an ordinary sentence iff at least one among the following conditions is satisfied :

(a) $\varphi \approx \perp$

(b) $\varphi \approx \uparrow\psi$

(c) $\varphi \approx (\psi \rightarrow \chi))$;

(d) $\varphi \approx \Diamond\psi$ and ψ is an ordinary sentence;

(e) $\varphi \approx \neg\psi$ and ψ is an ordinary sentence;

(f) $\varphi \approx (\psi \vee \chi)$ and both ψ and χ are ordinary sentences;

(g) $\varphi \approx (\psi \wedge \chi)$ and both ψ and χ are ordinary sentences;

By derived logical constants, the following result holds.

Theorem 4.4

Let φ and ψ be ordinary sentences. Then the following are also ordinary sentences: '$\top$', $\downarrow\varphi$, $(\varphi \leftrightarrow \psi)$, $\Box\varphi$

Proof
Omitted.

q.e.d.

Since the class $\mathscr{O}$ of the ordinary sentences is closed under all the logical constants except $\natural$ and $|$, we may use special meta-variables $\hat{\varphi}, \hat{\psi}, \hat{\chi}$, and so on for expressing schemas that refer to the class of ordinary sentences. A valid schema in which occur such meta-variables is meant to be valid in the sense that every sentence obtained by substitution of such meta-variables with ordinary sentences is true at every world in every model. We will call such schemas O-valid schemas.

We should prove that the set of ordinary sentences, as defined by definition 4.5, semantically coincides with the set of essentially two-valued sentences. The following result proves this.

Theorem 4.5

A sentence φ is an essentially two-valued sentence iff it is logically equivalent to an ordinary sentence.

Proof
If φ is an essentially two-valued sentence, so that for every model $\mathcal{M}$ and every world α in $\mathcal{M}$ it holds that either φ or $\neg\varphi$, then the sentence $\uparrow\varphi$ is logically equivalent to φ and it is an ordinary sentence.

If φ is logically equivalent to an ordinary sentence ψ, it is straightforward, by definition 4.5, definition 2.4 and theorem 2.1 that at every world α in every model $\mathcal{M}$, it holds: either ψ or $\neg\psi$, so that ψ is an essentially two-valued sentence.

q.e.d.

5. Normal Forms

In this section, we will prove first that every sentence of $\mathscr{L}$ is logically equivalent to a syntactically simple sentence. More precisely, we show that every sentence is logically equivalent to a sentence of the form $(\psi \mid \varphi)$, where both ψ and φ are essentially two-valued sentences, possibly containing occurrences of modal symbols. Second, we will prove that to every sentence φ of $\mathscr{L}$ we may effectively associate an S5-valid sentence φ' of the standard modal language $\mathscr{L}_2$ (which is a sub-language of $\mathscr{L}$) so that φ is valid iff φ' is S5-valid. To prove these results we need to prove several general results and to introduce the idea of the "normal quasi-classical form" and to prove the every essentially two-valued sentence is logically equivalent to a sentence in such a normal form.

Theorem 5.1

Let φ be a sentence of $\mathscr{L}$. At every world α in every model $\mathcal{M}$ it holds: iff

1. $\models_{\alpha}^{\mathcal{M}} \uparrow\varphi$ iff $\overline{\models}_{\alpha}^{\mathcal{M}} \uparrow\neg\varphi$
2. $\overline{\models}_{\alpha}^{\mathcal{M}} \uparrow\varphi$ iff $\models_{\alpha}^{\mathcal{M}} \uparrow\neg\varphi$

Proof
Omitted.

q.e.d.

Theorem 5.2

Double negation law. Let φ be a sentence of $\mathscr{L}$. The following equivalence holds: $\models \varphi \leftrightarrow \neg\neg\varphi$

Proof
Let $\mathcal{M}$ be a model and α any world in $\mathcal{M}$. Suppose $\models_{\alpha}^{\mathcal{M}} \varphi$. Then, by definition 2.4 $\overline{\models}_{\alpha}^{\mathcal{M}} \neg\varphi$ and also $\models_{\alpha}^{\mathcal{M}} \neg\neg\varphi$. Suppose that $\overline{\models}_{\alpha}^{\mathcal{M}} \varphi$. Then, by definition 2.4 $\models_{\alpha}^{\mathcal{M}} \neg\varphi$ and also $\overline{\models}_{\alpha}^{\mathcal{M}} \neg\neg\varphi$. Suppose that not $\models_{\alpha}^{\mathcal{M}} \varphi$ and not $\overline{\models}_{\alpha}^{\mathcal{M}} \varphi$. Then, by definition 2.4, not $\models_{\alpha}^{\mathcal{M}} \neg\varphi$ and not $\overline{\models}_{\alpha}^{\mathcal{M}} \neg\varphi$ and also not $\models_{\alpha}^{\mathcal{M}} \neg\neg\varphi$ and not $\overline{\models}_{\alpha}^{\mathcal{M}} \neg\neg\varphi$. So φ and $\neg\varphi$ have the same truth-value at α, so that, by theorem 2.1 $\models_{\alpha}^{\mathcal{M}} \varphi \leftrightarrow \neg\neg\varphi$. Since this holds for every world in every model, it holds that $\models \varphi \leftrightarrow \neg\neg\varphi$.

q.e.d.

5.1 Simple Tri-events

A simple conditional is a sentence of the form 'if A then C' where A and C are categorical sentences. Our language $\mathscr{L}$ contains two conditional constants: the particular two-valued material implication ('$\rightarrow$') and conditioning ('$|$'). Only the latter is assumed to represent those conditionals that pass the Ramsey-test. So, we characterise simple tri-events as those sentences of $\mathscr{L}$ having the form $\psi \mid \varphi$, where both ψ and φ do not contain any occurrence of the conditioning symbol '$|$' (nor any of the defined connectives defined in terms of '$|$'). This characterisation makes sense only if '$|$' cannot be defined in terms of the other primitive connectives. This is proved by the following result.

Theorem 5.3

There are a model $\mathcal{M}$ and two sentences φ and ψ of $\mathscr{L}$ containing only primitive symbols such that no sentence χ of $\mathscr{L}$ that does not contain any occurrence of the symbol '$|$' is $\mathcal{M}$-equivalent to $(\psi \mid \varphi)$.

Proof

Let φ and ψ be two sentences of $\mathscr{L}$ (containing only primitive symbols) and let $\mathcal{M}$ be a model. Let us assume, without loss of generality that:

(a) φ, ψ, and $\mathcal{M}$ are such that both φ and ψ are two valued in $\mathcal{M}$, ψ is not a $\mathcal{M}$–consequence of $\{\varphi\}$, φ is not a $\mathcal{M}$–consequence of $\{\psi\}$, both φ and ψ are $\mathcal{M}$–contingent sentences, both φ and ψ are $\mathcal{M}$–compatible sentences, so that

   (i) at each world α in $\mathcal{M}$ either $\models^{\mathcal{M}}_{\alpha} \varphi$ or $\overline{\models}^{\mathcal{M}}_{\alpha} \varphi$ and either $\models^{\mathcal{M}}_{\alpha} \psi$ or $\overline{\models}^{\mathcal{M}}_{\alpha} \psi$

   (ii) there are worlds α, β, γ, δ all in $\mathcal{M}$, such that $\models^{\mathcal{M}}_{\alpha} \varphi$, $\models^{\mathcal{M}}_{\beta} \psi$, $\overline{\models}^{\mathcal{M}}_{\gamma} \varphi$, and $\overline{\models}^{\mathcal{M}}_{\delta} \psi$;

   (iii) there is a world α in $\mathcal{M}$ at which $\models^{\mathcal{M}}_{\alpha} \varphi$ and $\overline{\models}^{\mathcal{M}}_{\alpha} \psi$;

   (iv) there is a world α in $\mathcal{M}$ at which $\overline{\models}^{\mathcal{M}}_{\alpha} \varphi$ and $\models^{\mathcal{M}}_{\alpha} \psi$;

   (v) there is a world β in $\mathcal{M}$ at which $\models^{\mathcal{M}}_{\beta} \varphi$ and $\models^{\mathcal{M}}_{\beta} \psi$.

(b) every atomic sentence of $\mathscr{L}$ is two-valued in $\mathcal{M}$.

Let χ be a sentence built using the primitives connectives of $\mathscr{L}$ except '|'. If χ does not contain any occurrence of atomic sentences, then either χ is $\mathcal{M}$–equivalent to '$\perp$' or χ is $\mathcal{M}$–equivalent to '$\neg\perp$'. In fact, the truth conditions of χ depend on no atomic sentence, so that χ has either no truth-value at every world or the same truth-value at all worlds. Moreover, by definition 2.1 χ is built using recursively the clauses (b)-(d) of such definition. By definition 4.5 χ is an ordinary sentence and therefore by theorem 4.5 χ is an essentially two valued sentence and *a fortiori* a two-valued sentence in $\mathcal{M}$. So, χ cannot be neither true nor false at every world in $\mathcal{M}$. In the case χ is $\mathcal{M}$–equivalent to '$\perp$', by condition (v) it holds that there is a world α in $\mathcal{M}$ such that $\models^{\mathcal{M}}_{\alpha} (\psi \mid \varphi)$, so that χ is not $\mathcal{M}$-equivalent to $(\psi \mid \varphi)$. In the case χ is $\mathcal{M}$–equivalent to '$\neg\perp$', by condition (iii) there is a world α in $\mathcal{M}$ at which it holds that $\overline{\models}^{\mathcal{M}}_{\alpha} (\psi \mid \varphi)$, so that, again, χ is not $\mathcal{M}$-equivalent to $(\psi \mid \varphi)$. Suppose now that at least one atomic sentence occurs in χ. Let $\{\omega_1, \ldots, \omega_n\}(1 \leq n)$ be the set of the atomic sentences occurring in χ. Since every atomic sentence ω_i $(1 \leq i \leq n)$ is, by conditions (b), two-valued in $\mathcal{M}$, it holds that $\uparrow\omega_i$ is $\mathcal{M}$–equivalent to ω_i. By definition 4.5 for every $i(1 \leq i \leq n)$ $\uparrow\omega_i$ is an ordinary sentence. By the same definition 4.5, χ is $\mathcal{M}$–equivalent to the sentence χ' obtained replacing in χ every occurrence of every $\uparrow\omega_i$ with ω_i. By theorem 4.5 χ' is essentially two-valued and χ, being $\mathcal{M}$–equivalent to χ', is two-valued in $\mathcal{M}$. Now, $(\psi \mid \varphi)$ is not two-valued, since by conditions (ii) and (iv) and 2.4 there is a world α at which neither $\models^{\mathcal{M}}_{\alpha} (\psi \mid \varphi)$ nor $\overline{\models}^{\mathcal{M}}_{\alpha} (\psi \mid \varphi)$. This proves that no sentence of $\mathscr{L}$ that do not contain occurrences of the symbol '|' may be $\mathcal{M}$–equivalent to $(\psi \mid \varphi)$.

q.e.d.

Ernest Adams developed (1975) a beautiful logic (APL) for these conditionals, albeit restricted to so-called *simple conditionals*. As anticipated above in section 1., our theory aims at extending Adams' theory to compounds of conditionals, providing for them a truth-conditional semantics. Since the present theory does not assume that conditionals are two-valued sentences, our task is not at odds with LTR. However, and we consider this is a very fundamental result, each compound of tri-events is logically equivalent to a simple tri-event (in the sense that they have the same truth conditions according to our semantics). Moreover, there is an effective way to associate a simple logically equivalent conditional to every tri-event.

Let us begin by defining the notion of simple tri-events properly. There are two definitions of this notion, one syntactical the other being semantical.

Definition 5.1 (Syntactically simple tri-events)

A sentence φ of $\mathscr{L}$ is said to be a syntactically simple tri-event iff $\varphi = (\psi \mid \chi)$ where ψ and χ are ordinary sentences.

Definition 5.2 (Semantically simple tri-events)

A sentence φ of $\mathscr{L}$ is said to be a semantically simple tri-event iff $\varphi = (\psi \mid \chi)$ and both ψ and χ are essentially two-valued sentences.

The following result, originally proved by de Finetti with respect to his truth-functional semantics (1995), which is kept in the present theory, shows that every sentence is logically equivalent to a semantically simple tri-event.

Theorem 5.4

Every sentence φ is logically equivalent to a semantically simple tri-event.

Proof

Let ψ be the sentence $(\uparrow\varphi \mid\updownarrow \varphi)$. We'll prove (a) that ψ is a semantically simple tri-event, and (b) that φ and ψ are logically equivalent.

(A) It suffices to prove that $\uparrow\varphi$ and $\updownarrow \varphi$ are ordinary sentences. $\uparrow\varphi$ is by definition 4.5 an ordinary sentence. $\updownarrow \varphi$ is by 2.2 an abbreviation of $(\uparrow\varphi \vee \uparrow\neg\varphi)$ that is the disjunction of two sentence that, by definition 4.5 are ordinary sentences and is therefore, by the same 4.5 an ordinary sentence.

(B) We prove (i) that $\{\varphi\} \models \psi$, and (ii) $\{\psi\} \models \varphi$.

(i) Suppose that in a model $\mathcal{M}$ and at the world α in $\mathcal{M}$ it holds that if φ. Then, by definition 2.4, $\models_\alpha^\mathcal{M} \uparrow\varphi$ and by theorem 2.1, $\models_\alpha^\mathcal{M}\updownarrow \varphi$. By definition 2.4, $\models_\alpha^\mathcal{M} (\uparrow\varphi \mid\updownarrow \varphi)$. Suppose now that not $\overline{\models}_\alpha^\mathcal{M} \varphi$ then, by theorem 2.1, either $\models_\alpha^\mathcal{M}\updownarrow \varphi$ and $\models_\alpha^\mathcal{M} \uparrow\varphi$ (so that as above $\models_\alpha^\mathcal{M} (\uparrow\varphi) \mid\updownarrow \varphi))$ or $\overline{\models}_\alpha^\mathcal{M}\updownarrow \varphi$, in which case, by definition 2.4, not $\overline{\models}_\alpha^\mathcal{M} (\uparrow\varphi \mid\updownarrow \varphi)$.

(ii) Suppose that at the world α in the model $\mathcal{M}$ it holds that $\models_\alpha^{\mathcal{M}} \psi$. Then $\models_\alpha^{\mathcal{M}} (\uparrow\varphi \mid\updownarrow \varphi)$. By definition 2.4 either both $\models_\alpha^{\mathcal{M}} \uparrow\varphi$ and $\models_\alpha^{\mathcal{M}}\updownarrow \varphi$ or, by theorem 3.2, ψ is valid. Now, if $\models_\alpha^{\mathcal{M}} \uparrow\varphi$ then $\models_\alpha^{\mathcal{M}} \varphi$. If $\models_{\mathcal{M}} \psi$ then it holds that $\models_\alpha^{\mathcal{M}} \psi$ at every world α. In this case, at any world α cannot be $\overline{\models}_\alpha^{\mathcal{M}} \varphi$ because, if so, would be that $\overline{\models}_\alpha^{\mathcal{M}} \uparrow\varphi$, being $\uparrow\varphi$ either true or false and not true by definition 2.4. In this case would be that $\models_\alpha^{\mathcal{M}}\updownarrow \varphi$, so that would be that $\overline{\models}_\alpha^{\mathcal{M}} (\uparrow\varphi \mid\updownarrow \varphi)$. But $\psi = \uparrow(\varphi \mid\updownarrow \varphi)$, so would be that $\overline{\models}_\alpha^{\mathcal{M}} \psi$, contrary the assumption that $\models_{\mathcal{M}} \psi$. Moreover, if $\models_{\mathcal{M}} \psi$ there is a world α in $\mathcal{M}$ at which it holds that $\models_\alpha^{\mathcal{M}} \varphi$. Otherwise it would be that $\overline{\models}_\alpha^{\mathcal{M}} \uparrow\varphi$ at every world α in $\mathcal{M}$ and $\models_\alpha^{\mathcal{M}} \psi$ at no world in $\mathcal{M}$.

Suppose that at the world α in the model $\mathcal{M}$ it holds that not $\overline{\models}_\alpha^{\mathcal{M}} \psi$. We have already proved that if $\models_\alpha^{\mathcal{M}} \psi$ then also $\models_\alpha^{\mathcal{M}} \varphi$. Suppose then that ψ is such that not$\models_\alpha^{\mathcal{M}} \psi$ and not $\overline{\models}_\alpha^{\mathcal{M}} \psi$. In this case $\overline{\models}_\alpha^{\mathcal{M}}\updownarrow \varphi$ and by definition 2.4, not $\models_\alpha^{\mathcal{M}} (\uparrow\varphi \mid\updownarrow \varphi)$ and not $\overline{\models}_\alpha^{\mathcal{M}} (\uparrow\varphi \mid\updownarrow \varphi)$, in which case would be not $\models_\alpha^{\mathcal{M}} \psi$ and not $\overline{\models}_\alpha^{\mathcal{M}} \psi$.

q.e.d.

Before proving the syntactical counterpart of theorem 5.4, we will prove some relevant results from which such a theorem follows quickly. Some of the following equivalences are based on the fact that the truth conditions that govern connectives may be made explicit by means of modal symbols and the unary symbol '↑', along the other connectives.

Theorem 5.5

The following equivalences hold:

1. $\models \uparrow(\varphi \vee \psi) \leftrightarrow ((\uparrow\varphi \vee \uparrow\psi) \vee (\neg\Diamond(\uparrow\neg\varphi \wedge \uparrow\neg\psi) \wedge (\Diamond\uparrow\varphi \vee \Diamond\uparrow\psi)))$
2. $\models \uparrow\neg(\varphi \vee \psi) \leftrightarrow (\uparrow\neg\varphi \wedge \uparrow\neg\psi)$
3. $\models \uparrow(\varphi \wedge \psi) \leftrightarrow (\uparrow\varphi \wedge \uparrow\psi)$
4. $\models \uparrow\neg(\varphi \wedge \psi) \leftrightarrow (\uparrow\neg\varphi \vee \uparrow\neg\psi \vee (\neg\Diamond(\uparrow\varphi \wedge \uparrow\psi) \wedge (\Diamond\uparrow\neg\varphi \vee \Diamond\uparrow\neg\psi)))$
5. $\models \uparrow(\psi \mid \varphi) \leftrightarrow ((\uparrow\varphi \wedge \uparrow\psi) \vee (\Diamond(\uparrow\varphi \wedge \uparrow\psi) \wedge \neg\Diamond(\uparrow\varphi \wedge \uparrow\neg\psi)))$
6. $\models \uparrow\neg(\psi \mid \varphi) \leftrightarrow \uparrow(\neg\psi \mid \varphi)$
7. $\models \uparrow\Diamond\varphi \leftrightarrow \Diamond\uparrow\varphi$
8. $\models \uparrow\neg\Diamond\varphi \leftrightarrow (\uparrow(\varphi \vee \neg\varphi) \wedge \neg\Diamond\uparrow\varphi)$
9. $\models \uparrow(\varphi \rightarrow \psi) \leftrightarrow (\uparrow\psi \vee (\neg\uparrow\varphi \wedge \neg\uparrow\neg\psi)) \vee \uparrow\neg\varphi)$
10. $\models \uparrow\neg(\varphi \rightarrow \psi) \leftrightarrow ((\uparrow\varphi \wedge \neg\uparrow\psi) \vee (\uparrow\neg\psi \wedge \neg\uparrow\neg\varphi))$

Proof

By definition 2.4 for every pair of sentences φ and ψ, every model $\mathcal{M}$ and every world α in $\mathcal{M}$, it holds that (i) $\models_\alpha^{\mathcal{M}} \uparrow\varphi$ iff $\models_\alpha^{\mathcal{M}} \varphi$, and that (ii) $\models_{\mathcal{M}} (\varphi \leftrightarrow$

ψ) amounts to the satisfaction of the two following conditions: (i) $\models^{\mathcal{M}}_{\alpha} \varphi$ iff $\models^{\mathcal{M}}_{\alpha} \psi$ and (ii) $\overline{\models}^{\mathcal{M}}_{\alpha} \varphi$ iff $\overline{\models}^{\mathcal{M}}_{\alpha} \psi$. Since in all the equivalences both the left side and the right side are ordinary sentences by definition 4.5, only the condition (i) needs to be proved. We prove:

1. $\models \uparrow(\varphi \vee \psi) \leftrightarrow ((\uparrow\varphi \vee \uparrow\psi) \vee (\neg\Diamond(\uparrow\neg\varphi \wedge \uparrow\neg\psi) \wedge (\Diamond\uparrow\varphi \vee \Diamond\uparrow\psi)))$

   (a) Suppose, first, that for some model $\mathcal{M}$ and some world α in $\mathcal{M}$, it holds that $\models^{\mathcal{M}}_{\alpha} (\varphi \vee \psi)$. We have to prove $\models^{\mathcal{M}}_{\alpha} ((\uparrow\varphi \vee \uparrow\psi) \vee (\neg\Diamond(\uparrow\neg\varphi \wedge \uparrow\neg\psi) \wedge (\Diamond\uparrow\varphi \vee \Diamond\uparrow\psi)))$. Suppose, *ab absurdo*, that this last sentence is false at some world α in $\mathcal{M}$. Since both $\chi = (\uparrow\varphi \vee \uparrow\psi)$ and $\zeta = (\neg\Diamond\uparrow\neg\varphi \wedge \uparrow\neg\psi) \wedge (\Diamond\uparrow\varphi \vee \Diamond\uparrow\psi)))$ are ordinary sentences, the entire sentence may be false only if holds that $\overline{\models}^{\mathcal{M}}_{\alpha} \chi$ and $\overline{\models}^{\mathcal{M}}_{\alpha} \zeta$. Now, $\overline{\models}^{\mathcal{M}}_{\alpha} \chi$ only if both $\overline{\models}^{\mathcal{M}}_{\alpha} \varphi$ and $\overline{\models}^{\mathcal{M}}_{\alpha} \psi$. $\overline{\models}^{\mathcal{M}}_{\alpha} \zeta$ only if at least one of its conjuncts is false at α. The first conjunct $(\neg\Diamond\uparrow\neg\varphi \wedge \uparrow\neg\psi)$ is false at α only if there is a world β in $\mathcal{M}$ at which both $\overline{\models}^{\mathcal{M}}_{\beta} \varphi$ and $\overline{\models}^{\mathcal{M}}_{\beta} \psi$ or not $\models^{\mathcal{M}}_{\gamma} \varphi$ and not $\models^{\mathcal{M}}_{\gamma} \psi$ at every world γ in $\mathcal{M}$. But in such a case, by definition 2.4, $\overline{\models}^{\mathcal{M}}_{\alpha} \uparrow(\varphi \vee \psi)$, which contradicts our hypothesis.

   (b) Suppose now that $((\uparrow\varphi\vee\uparrow\psi)\vee(\neg\Diamond(\uparrow\neg\varphi\wedge\uparrow\neg\psi)\wedge(\Diamond\uparrow\varphi\vee\Diamond\uparrow\psi)))$ is true at some world α in a certain model $\mathcal{M}$. So either $\chi = (\uparrow\varphi\vee\uparrow\psi)$ is true at α or $\zeta = (\neg\Diamond(\uparrow\neg\varphi\wedge\uparrow\neg\psi)\wedge(\Diamond\uparrow\varphi\vee\Diamond\uparrow\psi)))$ is true at α. In the first case, either $\models^{\mathcal{M}}_{\alpha} \varphi$ or $\models^{\mathcal{M}}_{\alpha} \psi$, so that, by definition 2.4, also $\models^{\mathcal{M}}_{\alpha} \uparrow(\varphi \vee \psi)$. In the second case, there is no world β in $\mathcal{M}$ such that $\overline{\models}^{\mathcal{M}}_{\beta} \varphi$ and $\overline{\models}^{\mathcal{M}}_{\beta} \psi$ and moreover there is a world γ at which either $\models^{\mathcal{M}}_{\gamma} \varphi$ or $\models^{\mathcal{M}}_{\gamma} \psi$. By definition 2.4, again, $\models^{\mathcal{M}}_{\alpha} \uparrow(\varphi \vee \psi)$.

2. $\models \uparrow\neg(\varphi \vee \psi) \leftrightarrow (\uparrow\neg\varphi \wedge \uparrow\neg\psi)$

   (a) Suppose that for some model $\mathcal{M}$ and some world α in $\mathcal{M}$, it holds that $\models^{\mathcal{M}}_{\alpha} \uparrow\neg(\varphi\vee\psi)$. By definition 2.4 it holds that $\overline{\models}^{\mathcal{M}}_{\alpha} (\varphi\vee\psi)$. By definition 2.4 it holds also that (j) either $\overline{\models}^{\mathcal{M}}_{\alpha} \varphi$ and $\overline{\models}^{\mathcal{M}}_{\alpha} \psi$ or (jj) for no world β in $\mathcal{M}$ it holds that either $\models^{\mathcal{M}}_{\beta} \varphi$ or $\models^{\mathcal{M}}_{\beta} \psi$ and there is a world γ in $\mathcal{M}$ such that it holds that $\overline{\models}^{\mathcal{M}}_{\gamma} \varphi$ and $\overline{\models}^{\mathcal{M}}_{\gamma} \psi$. In the case (j)it holds soon that $(\uparrow\neg\varphi \wedge \uparrow\neg\psi)$. In the case (jj) $(\varphi \vee \psi)$ is countervalid in $\mathcal{M}$ so that at every world γ in $\mathcal{M}$ it holds that $\overline{\models}^{\mathcal{M}}_{\gamma} \varphi$ and $\overline{\models}^{\mathcal{M}}_{\gamma} \psi$ and also $\models^{\mathcal{M}}_{\gamma} \uparrow\neg\varphi$ and $\models^{\mathcal{M}}_{\gamma} \uparrow\neg\psi$, from which it follows that $\models^{\mathcal{M}}_{\alpha} (\uparrow\neg\varphi \wedge \uparrow\neg\psi)$.

   (b) Suppose that for some model $\mathcal{M}$ and some world α in $\mathcal{M}$, it holds that $\models^{\mathcal{M}}_{\alpha} (\uparrow\neg\varphi \wedge \uparrow\neg\psi)$. By definition 2.4 $\overline{\models}^{\mathcal{M}}_{\alpha} \varphi$ and $\overline{\models}^{\mathcal{M}}_{\alpha} \psi$ and also $\overline{\models}^{\mathcal{M}}_{\alpha} (\varphi \vee \psi)$ and therefore $\models^{\mathcal{M}}_{\alpha} \uparrow\neg(\varphi \vee \psi)$.

3. $\models \uparrow(\varphi \wedge \psi) \leftrightarrow (\uparrow\varphi \wedge \uparrow\psi)$

(a) Suppose that for some model $\mathcal{M}$ and some world α in $\mathcal{M}$, it holds that $\models_\alpha^{\mathcal{M}} \{\uparrow(\varphi \wedge \psi)$. Suppose, *ab absurdo*, that not $\models_\alpha^{\mathcal{M}} (\uparrow\varphi \wedge \uparrow\psi)$. Since $(\uparrow\varphi \wedge \uparrow\psi)$ is an ordinary sentence, it follows that $\overline{\models}_\alpha^{\mathcal{M}} (\uparrow\varphi \wedge \uparrow\psi)$. Hence either $\overline{\models}_\alpha^{\mathcal{M}} \varphi$ or $\overline{\models}_\alpha^{\mathcal{M}} \psi$. In such a case $\overline{\models}_\alpha^{\mathcal{M}} (\varphi \wedge \psi)$ and therefore $\overline{\models}_\alpha^{\mathcal{M}} \uparrow(\varphi \wedge \psi)$, against the hypothesis.

(b) Suppose that $\models_\alpha^{\mathcal{M}} (\uparrow\varphi \wedge \uparrow\psi)$. Then $\models_\alpha^{\mathcal{M}} \varphi$ and $\models_\alpha^{\mathcal{M}} \psi$. By definition 2.4, $\models_\alpha^{\mathcal{M}} (\varphi \wedge \psi)$ and $\models_\alpha^{\mathcal{M}} \uparrow(\varphi \wedge \psi)$.

4. $\models \uparrow\neg(\varphi \wedge \psi) \leftrightarrow (\uparrow\neg\varphi \vee \uparrow\neg\psi \vee (\neg\Diamond(\uparrow\varphi \wedge \uparrow\psi) \wedge (\Diamond\uparrow\neg\varphi \vee \Diamond\uparrow\neg\psi)))$

Let ζ be $(\uparrow\neg\varphi \vee \uparrow\neg\psi \vee (\neg\Diamond(\uparrow\varphi \wedge \uparrow\psi) \wedge (\Diamond\uparrow\neg\varphi \vee \Diamond\uparrow\neg\psi)))$.

(a) Suppose that for some model $\mathcal{M}$ and some world α in $\mathcal{M}$, it holds that $\models_\alpha^{\mathcal{M}} \uparrow\neg(\varphi \wedge \psi)$. By definition 2.4 it holds that $\overline{\models}_\alpha^{\mathcal{M}} (\varphi \wedge \psi)$. By definition 2.4 it holds that either (j) $\overline{\models}_\alpha^{\mathcal{M}} \varphi$ or $\overline{\models}_\alpha^{\mathcal{M}} \psi$ or (jj) for no world β in $\mathcal{M}$ it holds that $\models_\alpha^{\mathcal{M}} \varphi$ and $\models_\alpha^{\mathcal{M}} \psi$ and at least one of the following conditions is satisfied: (k) there exists γ in $\mathcal{M}$ such that $\overline{\models}_\gamma^{\mathcal{M}} \varphi$, (kk) there exists γ in $\mathcal{M}$ such that $\overline{\models}_\gamma^{\mathcal{M}} \psi$. In the case (j) it holds, by definition 2.4, that $\overline{\models}_\alpha^{\mathcal{M}} (\varphi \wedge \psi)$ and also that $\models_\alpha^{\mathcal{M}} (\uparrow\neg\varphi \vee \uparrow\neg\psi)$ and also ζ. In the case (jj), at very world β $\overline{\models}_\beta^{\mathcal{M}} (\varphi \wedge \psi)$ so that, by definition 2.4 $\models_\alpha^{\mathcal{M}} \neg\Diamond(\uparrow\varphi \wedge \uparrow\psi)$. In this case it holds that $\models_\alpha^{\mathcal{M}} \zeta$.

(b) Suppose that for some model $\mathcal{M}$ and some world α in $\mathcal{M}$, it holds that $\models_\alpha^{\mathcal{M}} \zeta$. ζ is a disjunction of sub-sentences that, by definition 2.4 are either true or false at α. Therefore, either (j) $\models_\alpha^{\mathcal{M}} \uparrow\neg\varphi$ or (jj) $\models_\alpha^{\mathcal{M}} \uparrow\neg\psi$ or (jjj) $\models_\alpha^{\mathcal{M}} (\neg\Diamond(\uparrow\varphi \wedge \uparrow\psi) \wedge (\Diamond\uparrow\neg\varphi \vee \Diamond\uparrow\neg\psi))$. In the case (j) and (jj), ir follows from definition 2.4 that $\models_\alpha^{\mathcal{M}} \uparrow\neg(\varphi \wedge \psi)$. In the case (jjj) for every world β in $\mathcal{M}$ it holds that $\overline{\models}_\beta^{\mathcal{M}} (\varphi \wedge \psi)$ and therefore that $\overline{\models}_\alpha^{\mathcal{M}} (\varphi \wedge \psi)$. From definition 2.4 it follows that $\models_\alpha^{\mathcal{M}} \uparrow\neg(\varphi \wedge \psi)$.

5. $\models \uparrow(\psi \mid \varphi) \leftrightarrow ((\uparrow\varphi \wedge \uparrow\psi) \vee (\Diamond(\uparrow\varphi \wedge \uparrow\psi) \wedge \neg\Diamond(\uparrow\varphi \wedge \uparrow\neg\psi)))$

(a) Suppose, that for some model $\mathcal{M}$ and some world α in $\mathcal{M}$, it holds that $\models_\alpha^{\mathcal{M}} \uparrow(\psi \mid \varphi)$. By definition 2.4, either (j) both $\models_\alpha^{\mathcal{M}} \varphi$ and $\models_\alpha^{\mathcal{M}} \psi$ or (jj) there is a world β such that $\models_\beta^{\mathcal{M}} \varphi$ and $\models_\beta^{\mathcal{M}} \psi$ but there is no world γ such that $\models_\gamma^{\mathcal{M}} \varphi$ and $\overline{\models}_\gamma^{\mathcal{M}} \psi$. Now if (j) is satisfied then it holds that $\models_\alpha^{\mathcal{M}} (\uparrow\varphi \wedge \uparrow\psi)$, so that $\models_\alpha^{\mathcal{M}} \Diamond(\uparrow\varphi \wedge \uparrow\psi)$. If (jj) is satisfied then it holds that $\models_\beta^{\mathcal{M}} (\uparrow\varphi \wedge \uparrow\psi)$ for some β and for every γ it holds that not $\models_\gamma^{\mathcal{M}} (\uparrow\varphi \wedge \uparrow\neg\psi)$, so that $\models_\alpha^{\mathcal{M}} \neg\Diamond(\uparrow\varphi \wedge \uparrow\neg\psi)$. It follows that $\models_\alpha^{\mathcal{M}} ((\uparrow\varphi \wedge \uparrow\psi) \vee (\Diamond(\uparrow\varphi \wedge \uparrow\psi) \wedge \neg\Diamond(\uparrow\varphi \wedge \uparrow\neg\psi)))$ as desired.

(b) If $\models^{\mathcal{M}}_{\alpha}$ $((\uparrow\varphi \wedge \uparrow\psi) \vee (\Diamond(\uparrow\varphi \wedge \uparrow\psi) \wedge \neg\Diamond(\uparrow\varphi \wedge \uparrow\neg\psi)))$, since all subformulas of this sentence are ordinary sentences either (k) $\models^{\mathcal{M}}_{\alpha}$ $((\uparrow\varphi \wedge \uparrow\psi)$ or (kk) or $\models^{\mathcal{M}}_{\alpha}$ $(\Diamond(\uparrow\varphi \wedge \uparrow\psi) \wedge \neg\Diamond(\uparrow\varphi \wedge \uparrow\neg\psi))$ In the case (k), by definition 2.4, $\models^{\mathcal{M}}_{\alpha}$ $\uparrow(\psi \mid \varphi)$. In the case (kk), it holds that $\models^{\mathcal{M}}_{\alpha}$ $\Diamond(\uparrow\varphi \wedge \uparrow\psi)$ and $\models^{\mathcal{M}}_{\alpha}$ $\neg\Diamond(\uparrow\varphi \wedge \uparrow\neg\psi)$ and hence, by definition 2.4, $\models^{\mathcal{M}}_{\alpha}$ $\uparrow(\psi \mid \varphi)$.

6. $\models \uparrow\neg(\psi \mid \varphi) \leftrightarrow \uparrow(\neg\psi \mid \varphi)$

(a) Suppose that for some model $\mathcal{M}$ and some world α in $\mathcal{M}$, it holds that $\models^{\mathcal{M}}_{\alpha}$ $\uparrow\neg(\psi \mid \varphi)$. By definition 2.4 it holds that $\overline{\models}^{\mathcal{M}}_{\alpha}$ $(\psi \mid \varphi)$. But this entails that either (a) $\models^{\mathcal{M}}_{\alpha}$ φ and $\overline{\models}^{\mathcal{M}}_{\alpha}$ ψ or (b) there is a world β in $\mathcal{M}$ such that $\models^{\mathcal{M}}_{\beta}$ φ and $\overline{\models}^{\mathcal{M}}_{\beta}$ ψ, and for every world γ in $\mathcal{M}$ not $\models^{\mathcal{M}}_{\gamma}$ φ and not $\models^{\mathcal{M}}_{\gamma}$ ψ. In the case (a) $\models^{\mathcal{M}}_{\alpha}$ $\neg\psi$, so that $\models^{\mathcal{M}}_{\alpha}$ $\uparrow(\neg\psi \mid \varphi)$. In the case (b) $\models^{\mathcal{M}}_{\beta}$ $\neg\psi$ not $\overline{\models}^{\mathcal{M}}_{\gamma}$ $\neg\psi$, so that, by definition 2.4 $\models^{\mathcal{M}}_{\alpha}$ $(\psi \mid \varphi)$.

(b) Suppose that for some model $\mathcal{M}$ and some world α in $\mathcal{M}$, it holds that $\models^{\mathcal{M}}_{\alpha}$ $\uparrow(\neg\psi \mid \varphi)$. By point 3 above, it holds that $\models^{\mathcal{M}}_{\alpha}$ $((\uparrow\varphi \wedge \uparrow\neg\psi) \vee (\Diamond(\uparrow\varphi \wedge \uparrow\neg\psi) \wedge \neg\Diamond(\uparrow\varphi \wedge \uparrow\neg\neg\psi)))$. Either $\models^{\mathcal{M}}_{\alpha}$ $(\uparrow\varphi \wedge \uparrow\neg\psi)$ or $\models^{\mathcal{M}}_{\alpha}$ $(\Diamond(\uparrow\varphi \wedge \uparrow\neg\psi) \wedge \neg\Diamond(\uparrow\varphi \wedge \uparrow\neg\neg\psi))$. In the first case, by definition 2.4 $\models^{\mathcal{M}}_{\alpha}$ $\uparrow\neg(\psi \mid \varphi)$. In the second case, there is a world β in $\mathcal{M}$ such that $\models^{\mathcal{M}}_{\beta}$ $(\uparrow\varphi \wedge \uparrow\neg\psi)$ and for no world γ in $\mathcal{M}$ it holds that $\models^{\mathcal{M}}_{\gamma}$ $(\uparrow\varphi \wedge \uparrow\neg\neg\psi))$. Considering that by theorem 5.2 $\neg\neg\psi$ is logically equivalent to ψ there is no world γ in $\mathcal{M}$ at which $\models^{\mathcal{M}}_{\gamma}$ $(\uparrow\varphi \wedge \uparrow\psi)$. By definition 2.4 it holds that $\overline{\models}^{\mathcal{M}}_{\alpha}$ $\uparrow(\psi \mid \varphi)$ so that, by theorem 5.1, $\models^{\mathcal{M}}_{\alpha}$ $\uparrow\neg(\psi \mid \varphi)$

7. $\models \uparrow\Diamond\varphi \leftrightarrow \Diamond\uparrow\varphi$

(a) Suppose that for some model $\mathcal{M}$ and some world α in $\mathcal{M}$, it holds that $\models^{\mathcal{M}}_{\alpha}$ $\uparrow\Diamond\varphi$. By definition 2.4, it follows that $\models^{\mathcal{M}}_{\alpha}$ $\Diamond\varphi$. By definition 2.4, there is a world β such that $\models^{\mathcal{M}}_{\beta}$ φ and also $\models^{\mathcal{M}}_{\beta}$ $\uparrow\varphi$. It follows from definition 2.4 that $\models^{\mathcal{M}}_{\alpha}$ $\Diamond\uparrow\varphi$.

(b) Suppose that for some model $\mathcal{M}$ and some world α in $\mathcal{M}$, it holds that $\models^{\mathcal{M}}_{\alpha}$ $\Diamond\uparrow\varphi$. By definition 2.4, there is a world β such that $\models^{\mathcal{M}}_{\beta}$ $\uparrow\varphi$ and also $\models^{\mathcal{M}}_{\beta}$ φ. It follows $\models^{\mathcal{M}}_{\alpha}$ $\Diamond\varphi$ and also $\models^{\mathcal{M}}_{\alpha}$ $\uparrow\Diamond\varphi$

8. $\models \uparrow\neg\Diamond\varphi \leftrightarrow (\uparrow(\varphi \vee \neg\varphi) \wedge \neg\Diamond\uparrow\varphi)$

(a) Suppose that for some model $\mathcal{M}$ and some world α in $\mathcal{M}$, it holds that $\models^{\mathcal{M}}_{\alpha}$ $\uparrow\neg\Diamond\varphi$. By definition 2.4, it holds that $\overline{\models}^{\mathcal{M}}_{\alpha}$ $\Diamond\varphi$, so that $\overline{\models}_{\mathcal{M}}$ φ. This result, in turn, entails that φ is two-valued. Hence $\uparrow(\varphi \vee \neg\varphi)$. Moreover, $\models^{\mathcal{M}}_{\alpha}$ $\neg\Diamond\uparrow\varphi$. By definition 2.4 it holds that $\models^{\mathcal{M}}_{\alpha}$ $(\uparrow(\varphi \vee \neg\varphi) \wedge \neg\Diamond\uparrow\varphi)$.

(b) Suppose that for some model $\mathcal{M}$ and some world α in $\mathcal{M}$, it holds that $\models^{\mathcal{M}}_{\alpha} (\uparrow(\varphi \vee \neg\varphi) \wedge \neg\Diamond\uparrow\varphi)$. It follows that $\models^{\mathcal{M}}_{\alpha} \uparrow(\varphi \vee \neg\varphi)$ and $\models^{\mathcal{M}}_{\alpha} \neg\Diamond\uparrow\varphi$. From $\models^{\mathcal{M}}_{\alpha} \uparrow(\varphi \vee \neg\varphi)$ it follows that either $\models^{\mathcal{M}}_{\alpha} \varphi$ or $\overline{\models}^{\mathcal{M}}_{\alpha} \varphi$ and from that and $\models^{\mathcal{M}}_{\alpha} \neg\Diamond\uparrow\varphi$ it follows that $\overline{\models}^{\mathcal{M}}_{\alpha} \varphi$. Since this holds for every world β in $\mathcal{M}$, it holds that $\models^{\mathcal{M}}_{\alpha} \uparrow\neg\Diamond\varphi$.

9. $\models \uparrow(\varphi \to \psi) \leftrightarrow (\uparrow\psi \vee (\neg\uparrow\varphi \wedge \neg\uparrow\neg\psi) \vee \uparrow\neg\varphi)$
Let ζ be $(\uparrow\psi \vee (\neg\uparrow\varphi \wedge \neg\uparrow\neg\psi) \vee \uparrow\neg\varphi)$.

(a) Suppose that for some model $\mathcal{M}$ and some world α in $\mathcal{M}$, it holds that $\models^{\mathcal{M}}_{\alpha} \uparrow(\varphi \to \psi)$. This entails by definition 2.4 that $\models^{\mathcal{M}}_{\alpha} (\varphi \to \psi)$. This, in turn, entails that either $\models^{\mathcal{M}}_{\alpha} \psi$ or $\overline{\models}^{\mathcal{M}}_{\alpha} \varphi$ or both not $\models^{\mathcal{M}}_{\alpha} \varphi$ and not $\overline{\models}^{\mathcal{M}}_{\alpha} \psi$. If $\models^{\mathcal{M}}_{\alpha} \psi$ then $\models^{\mathcal{M}}_{\alpha} \uparrow\psi$, so that it also holds that $\models^{\mathcal{M}}_{\alpha} \zeta$. If $\overline{\models}^{\mathcal{M}}_{\alpha} \varphi$ then $\models^{\mathcal{M}}_{\alpha} \neg\varphi$ which entails $\models^{\mathcal{M}}_{\alpha} \uparrow\neg\varphi$, so that so that it also holds that $\models^{\mathcal{M}}_{\alpha} \zeta$. If neither $\models^{\mathcal{M}}_{\alpha} \varphi$ nor $\overline{\models}^{\mathcal{M}}_{\alpha} \psi$ it holds that $\models^{\mathcal{M}}_{\alpha} (\neg\uparrow\varphi \wedge \neg\uparrow\neg\psi)$, so that so that it also holds that $\models^{\mathcal{M}}_{\alpha} \zeta$.

(b) Suppose that for some model $\mathcal{M}$ and some world α in $\mathcal{M}$, it holds that $\models^{\mathcal{M}}_{\alpha} \zeta$. This entails, by definition 2.4 that either $\models^{\mathcal{M}}_{\alpha} \psi$ or $\overline{\models}^{\mathcal{M}}_{\alpha} \varphi$ or neither $\models^{\mathcal{M}}_{\alpha} \varphi$ nor $\overline{\models}^{\mathcal{M}}_{\alpha} \psi$. By definition 2.4, each of these conditions entail that $\models^{\mathcal{M}}_{\alpha} (\varphi \to \psi)$ and also $\models^{\mathcal{M}}_{\alpha} \uparrow(\varphi \to \psi)$.

10. $\models \uparrow\neg(\varphi \to \psi) \leftrightarrow ((\uparrow\varphi \wedge \neg\uparrow\psi) \vee (\uparrow\neg\psi \wedge \neg\uparrow\neg\varphi))$
Let ζ be $((\uparrow\varphi \wedge \neg\uparrow\psi) \vee (\uparrow\neg\psi \wedge \neg\uparrow\neg\varphi))$.

(a) Suppose that for some model $\mathcal{M}$ and some world α in $\mathcal{M}$, it holds that $\models^{\mathcal{M}}_{\alpha} \uparrow\neg(\varphi \to \psi)$. It follows that $\overline{\models}^{\mathcal{M}}_{\alpha} (\varphi \to \psi)$. By definition 2.4, either $\models^{\mathcal{M}}_{\alpha} \varphi$ and not $\models^{\mathcal{M}}_{\alpha} \psi$ or $\overline{\models}^{\mathcal{M}}_{\alpha} \psi$ and not $\overline{\models}^{\mathcal{M}}_{\alpha} \varphi$. Now, if $\models^{\mathcal{M}}_{\alpha} \varphi$ then $\models^{\mathcal{M}}_{\alpha} \uparrow\varphi$ and if not $\models^{\mathcal{M}}_{\alpha} \psi$ then $\models^{\mathcal{M}}_{\alpha} \neg\uparrow\psi$, so that it holds that $\models^{\mathcal{M}}_{\alpha} (\uparrow\varphi \wedge \neg\uparrow\psi)$. On the other hand, if $\overline{\models}^{\mathcal{M}}_{\alpha} \psi$ then $\models^{\mathcal{M}}_{\alpha} \uparrow\neg\psi$ ad if not $\overline{\models}^{\mathcal{M}}_{\alpha} \varphi$ then $\models^{\mathcal{M}}_{\alpha} \neg\uparrow\neg\varphi$, so that it holds that $\models^{\mathcal{M}}_{\alpha} (\uparrow\neg\psi \wedge \neg\uparrow\neg\varphi)$. So also $\models^{\mathcal{M}}_{\alpha} \zeta$.

(b) Suppose that for some model $\mathcal{M}$ and some world α in $\mathcal{M}$, it holds that $\models^{\mathcal{M}}_{\alpha} \zeta$ so that either $\models^{\mathcal{M}}_{\alpha} (\uparrow\varphi \wedge \neg\uparrow\psi)$ or $\models^{\mathcal{M}}_{\alpha} (\uparrow\neg\psi \wedge \neg\uparrow\neg\varphi)$. In the first case it holds that $\models^{\mathcal{M}}_{\alpha} \varphi$ and not $\models^{\mathcal{M}}_{\alpha} \psi$. By definition 2.4, $\overline{\models}^{\mathcal{M}}_{\alpha} (\varphi \to \psi)$ and therefore $\models^{\mathcal{M}}_{\alpha} \uparrow\neg(\varphi \to \psi)$. In the second case, $\overline{\models}^{\mathcal{M}}_{\alpha} \psi$ and not $\overline{\models}^{\mathcal{M}}_{\alpha} \varphi$. By definition 2.4, $\overline{\models}^{\mathcal{M}}_{\alpha} (\varphi \to \psi)$ and therefore $\models^{\mathcal{M}}_{\alpha} \uparrow\neg(\varphi \to \psi)$. So, in every case it holds that $\models^{\mathcal{M}}_{\alpha} \uparrow\neg(\varphi \to \psi)$.

q.e.d.

Theorem 5.6

(De Morgan Laws). Let φ and ψ to sentences of Let φ be a sentence of $\mathscr{L}$. The following equivalences hold:

1. $\models (\varphi \wedge \psi) \leftrightarrow \neg(\neg\varphi \vee \neg\psi)$
2. $\models (\varphi \vee \psi) \leftrightarrow \neg(\neg\varphi \wedge \neg\psi)$
3. $\models \neg(\varphi \wedge \psi) \leftrightarrow (\neg\varphi \vee \neg\psi)$
4. $\models \neg(\varphi \vee \psi) \leftrightarrow (\neg\varphi \wedge \neg\psi)$

Proof

1. $\models (\varphi \wedge \psi) \leftrightarrow \neg(\neg\varphi \vee \neg\psi)$
   (a) Suppose that for a certain world α in the model $\mathcal{M}$ it holds that $\models^{\mathcal{M}}_{\alpha} (\varphi \wedge \psi)$. By definition 2.4 it holds that $\models^{\mathcal{M}}_{\alpha} \uparrow(\varphi \wedge \psi)$. By theorem 5.5 it holds that $\models^{\mathcal{M}}_{\alpha} (\uparrow\varphi \wedge \uparrow\psi)$ and therefore $\models^{\mathcal{M}}_{\alpha} \varphi$ and $\models^{\mathcal{M}}_{\alpha} \psi$. It follows that $\overline{\models}^{\mathcal{M}}_{\alpha} \neg\varphi)$ and $\overline{\models}^{\mathcal{M}}_{\alpha} \neg\psi$. By definition 2.4 it holds that $\overline{\models}^{\mathcal{M}}_{\alpha} (\neg\varphi \vee \neg\psi)$ and, finally, that $\models^{\mathcal{M}}_{\alpha} \neg(\neg\varphi \vee \neg\psi)$.
   (b) Suppose that for a certain world α in the model $\mathcal{M}$ it holds that $\models^{\mathcal{M}}_{\alpha} \neg(\neg\varphi \vee \neg\psi)$. It holds that $\models^{\mathcal{M}}_{\alpha} \uparrow\neg(\neg\varphi \vee \neg\psi)$. By theorem 5.5 it holds that $\models^{\mathcal{M}}_{\alpha} (\uparrow\neg\neg\varphi \wedge \uparrow\neg\neg\psi$. By theorem 5.2 it holds that $\models^{\mathcal{M}}_{\alpha} (\uparrow\varphi \wedge \uparrow\psi$. By theorem 5.5 it holds $\models^{\mathcal{M}}_{\alpha} \uparrow(\varphi \wedge \psi)$. Therefore, by definition 2.4 it holds that $\models^{\mathcal{M}}_{\alpha} (\varphi \wedge \psi)$.
   (c) Suppose that for a certain world α in the model $\mathcal{M}$ it holds that $\overline{\models}^{\mathcal{M}}_{\alpha} (\varphi \wedge \psi)$. By definition 2.4, there are three exhaustive cases: (j) $\models^{\mathcal{M}}_{\alpha} \uparrow\neg\varphi$, (jj) $\models^{\mathcal{M}}_{\alpha} \uparrow\neg\psi$, (jjj) it holds that either that (k) for every world β in $\mathcal{M}$, either not $\models^{\mathcal{M}}_{\beta} \varphi$ or not $\models^{\mathcal{M}}_{\beta} \psi$, or (kk) there is a world γ in $\mathcal{M}$ such that it holds that either $\overline{\models}^{\mathcal{M}}_{\gamma} \varphi$ or $\overline{\models}^{\mathcal{M}}_{\gamma} \psi$. In the case (j) it holds that $\overline{\models}^{\mathcal{M}}_{\alpha} \varphi$ and therefore $\models^{\mathcal{M}}_{\alpha} \neg\varphi$ and also $\models^{\mathcal{M}}_{\alpha} (\neg\varphi \vee \neg\psi)$ and therefore $\overline{\models}^{\mathcal{M}}_{\alpha} \neg(\neg\varphi \vee \neg\psi)$. In the case (jj) it holds that $\overline{\models}^{\mathcal{M}}_{\alpha} \psi$ and therefore $\models^{\mathcal{M}}_{\alpha} \neg\psi$ and also $\models^{\mathcal{M}}_{\alpha} (\neg\varphi \vee \neg\psi)$ and therefore $\overline{\models}^{\mathcal{M}}_{\alpha} \neg(\neg\varphi \vee \neg\psi)$. In the case (jjj) $\overline{\models}_{\mathcal{M}} (\varphi \wedge \psi)$ so that $\models_{\mathcal{M}} \neg(\varphi \wedge \psi)$. So at every world β in $\mathcal{M}$ it holds that either $\models^{\mathcal{M}}_{\beta} \neg\varphi$ or $\models^{\mathcal{M}}_{\beta} \neg\psi$. It follows that $\models^{\mathcal{M}}_{\beta} (\neg\varphi \vee \psi)$ and therefore that $\overline{\models}^{\mathcal{M}}_{\beta} \neg(\neg\varphi \vee \psi)$.
   (d) Suppose that for a certain world α in the model $\mathcal{M}$ it holds that $\overline{\models}^{\mathcal{M}}_{\alpha} \neg(\neg\varphi \vee \neg\psi)$. By theorem 5.5 it holds that $\models^{\mathcal{M}}_{\alpha} (\neg\varphi \vee \neg\psi)$. By definition 2.4, there are two exhaustive cases: (j) either $\overline{\models}^{\mathcal{M}}_{\alpha} \varphi$ or $\overline{\models}^{\mathcal{M}}_{\alpha} \psi$, and (jj) in no world β in $\mathcal{M}$ it holds that $\models^{\mathcal{M}}_{\beta} \varphi$ and $\models^{\mathcal{M}}_{\beta} \psi$ and there is a world γ in $\mathcal{M}$ such that either $\overline{\models}^{\mathcal{M}}_{\gamma} \varphi$ or $\overline{\models}^{\mathcal{M}}_{\alpha} \psi$. In

the case (j), by definition 2.4 it holds that $\overline{\models}^{\mathcal{M}}_{\alpha} (\varphi \wedge \psi)$. In the case (jj) at every world β in $\mathcal{M}$ it holds that $\overline{\models}^{\mathcal{M}}_{\beta} (\varphi \wedge \psi)$ and therefore also $\overline{\models}^{\mathcal{M}}_{\alpha} (\varphi \wedge \psi)$.

2. $\models (\varphi \vee \psi) \leftrightarrow \neg(\neg\varphi \wedge \neg\psi)$

   (a) Suppose that for a certain world α in the model $\mathcal{M}$ it holds that $\models^{\mathcal{M}}_{\alpha} (\varphi \vee \psi)$. By definition 2.4 there are two cases: (j) either $\models^{\mathcal{M}}_{\alpha} \varphi$ or $\models^{\mathcal{M}}_{\alpha} \psi$, and (jj) at every world β in $\mathcal{M}$ it does not hold that $\overline{\models}^{\mathcal{M}}_{\beta} \varphi$ and $\overline{\models}^{\mathcal{M}}_{\alpha} \psi$ and there is a world γ in $\mathcal{M}$ such that it holds that either $\models^{\mathcal{M}}_{\gamma} \varphi$ or $\models^{\mathcal{M}}_{\gamma} \psi$. In the case (j) it holds that either $\overline{\models}^{\mathcal{M}}_{\alpha} \neg\varphi$ or $\overline{\models}^{\mathcal{M}}_{\alpha} \neg\psi$. In this case it holds that $\overline{\models}^{\mathcal{M}}_{\alpha} (\neg\varphi \wedge \neg\psi)$ and therefore $\models^{\mathcal{M}}_{\alpha} \neg(\neg\varphi \wedge \neg\psi)$. In the case (jj) it holds that $\overline{\models}_{\mathcal{M}} (\neg\varphi \wedge \neg\psi)$, so that $\models_{\mathcal{M}} \neg(\neg\varphi \wedge \neg\psi)$ and, in particular, $\models^{\mathcal{M}}_{\alpha} \neg(\neg\varphi \wedge \neg\psi)$.

   (b) Suppose that for a certain world α in the model $\mathcal{M}$ it holds that $\models^{\mathcal{M}}_{\alpha} \neg(\neg\varphi \wedge \neg\psi)$. By definition 2.4, there are two cases (j) either $\overline{\models}^{\mathcal{M}}_{\alpha} \neg\varphi$ or $\overline{\models}^{\mathcal{M}}_{\alpha} \neg\psi$ and (jj) at no world β in $\mathcal{M}$ it holds that at the same time $\models^{\mathcal{M}}_{\beta} \neg\varphi$ and $\models^{\mathcal{M}}_{\alpha} \neg\psi$ and there is a world γ in $\mathcal{M}$ such that either $\overline{\models}^{\mathcal{M}}_{\gamma} \neg\varphi$ or $\overline{\models}^{\mathcal{M}}_{\gamma} \neg\psi$. In the case (j) it holds that either $\models^{\mathcal{M}}_{\alpha} \varphi$ or $\models^{\mathcal{M}}_{\alpha} \psi$ and therefore it holds that $\models^{\mathcal{M}}_{\alpha} (\varphi \vee \psi)$. In the case (jj) $\models_{\mathcal{M}} (\varphi \vee \psi)$ and therefore $\models^{\mathcal{M}}_{\alpha} (\varphi \vee \psi)$.

   (c) Suppose that for a certain world α in the model $\mathcal{M}$ it holds that $\overline{\models}^{\mathcal{M}}_{\alpha} (\varphi \vee \psi)$. By definition 2.4, it holds that $\models^{\mathcal{M}}_{\alpha} \neg(\varphi \vee \psi)$ and by theorem 5.5 $\models^{\mathcal{M}}_{\alpha} (\neg\varphi \wedge \neg\psi)$. It follows that $\overline{\models}^{\mathcal{M}}_{\alpha} \neg(\neg\varphi \wedge \neg\psi)$.

   (d) Suppose that for a certain world α in the model $\mathcal{M}$ it holds that $\overline{\models}^{\mathcal{M}}_{\alpha} \neg(\neg\varphi \wedge \neg\psi)$. By definition 2.4 it holds that $\models^{\mathcal{M}}_{\alpha} (\neg\varphi \wedge \neg\psi)$ and therefore $\models^{\mathcal{M}}_{\alpha} \uparrow(\neg\varphi \wedge \neg\psi)$. By theorem 5.5 it holds that $\models^{\mathcal{M}}_{\alpha}$ $(\uparrow\neg\varphi \wedge \uparrow\neg\psi$. It follows from definition 2.4 that $\overline{\models}^{\mathcal{M}}_{\alpha} \uparrow(\varphi \vee \psi)$ and also $\overline{\models}^{\mathcal{M}}_{\alpha} (\varphi \vee \psi)$.

3. $\models \neg(\varphi \wedge \psi) \leftrightarrow (\neg\varphi \vee \neg\psi)$

   (a) Suppose that for a certain world α in the model $\mathcal{M}$ it holds that $\models^{\mathcal{M}}_{\alpha} \neg(\varphi \wedge \psi)$. Then, by definition 2.4, it holds that $\overline{\models}^{\mathcal{M}}_{\alpha} (\varphi \wedge \psi)$. By point (1) it holds that $\overline{\models}^{\mathcal{M}}_{\alpha} \neg(\neg\varphi \vee \neg\psi)$ and therefore $\models^{\mathcal{M}}_{\alpha} (\neg\varphi \vee \neg\psi)$.

   (b) Suppose that for a certain world α in the model $\mathcal{M}$ it holds that $\models^{\mathcal{M}}_{\alpha} (\neg\varphi \vee \neg\psi)$. By definition 2.4, it holds that $\overline{\models}^{\mathcal{M}}_{\alpha} \neg(\neg\varphi \vee \neg\psi)$ By what proved under (1) it holds that $\overline{\models}^{\mathcal{M}}_{\alpha} (\varphi \wedge \psi)$, and therefore $\models^{\mathcal{M}}_{\alpha} \neg(\varphi \wedge \psi)$.

(c) Suppose that for a certain world α in the model $\mathcal{M}$ it holds that $\overline{\models}^{\mathcal{M}}_{\alpha} \neg(\varphi \wedge \psi)$. By definition 2.4, it holds that $\models^{\mathcal{M}}_{\alpha} (\varphi \wedge \psi)$. By what has been proved under (1) it holds that $\models^{\mathcal{M}}_{\alpha} \neg(\neg\varphi \vee \neg\psi)$, and therefore $\overline{\models}^{\mathcal{M}}_{\alpha} \neg\varphi \vee \neg\psi)$.

(d) Suppose that for a certain world α in the model $\mathcal{M}$ it holds that $\overline{\models}^{\mathcal{M}}_{\alpha} (\neg\varphi \vee \neg\psi)$. By definition 2.4, it holds that $\models^{\mathcal{M}}_{\alpha} \neg(\neg\varphi \vee \neg\psi)$. By what has been proved under (1) it holds that $\models^{\mathcal{M}}_{\alpha} (\varphi \wedge \psi$. It follows that $\overline{\models}^{\mathcal{M}}_{\alpha} \neg(\varphi \wedge \psi)$.

4. $\models \neg(\varphi \vee \psi) \leftrightarrow (\neg\varphi \wedge \neg\psi)$

(a) Suppose that for a certain world α in the model $\mathcal{M}$ it holds that $\models^{\mathcal{M}}_{\alpha} \neg(\varphi \vee \psi)$. By definition 2.4, it holds that $\overline{\models}^{\mathcal{M}}_{\alpha} (\varphi \vee \psi)$. By what has been proved under (2) it holds that $\overline{\models}^{\mathcal{M}}_{\alpha} \neg(\neg\varphi \wedge \neg\psi)$ and therefore that $\models^{\mathcal{M}}_{\alpha} (\neg\varphi \wedge \neg\psi)$.

(b) Suppose that for a certain world α in the model $\mathcal{M}$ it holds that $\models^{\mathcal{M}}_{\alpha} (\neg\varphi \wedge \neg\psi)$. By definition 2.4, it holds that $\overline{\models}^{\mathcal{M}}_{\alpha} \neg(\neg\varphi \wedge \neg\psi)$. By what has been proved under (2) it holds that $\overline{\models}^{\mathcal{M}}_{\alpha} (\varphi \vee \psi)$ and therefore that $\models^{\mathcal{M}}_{\alpha} \neg(\varphi \vee \psi)$.

(c) Suppose that for a certain world α in the model $\mathcal{M}$ it holds that $\overline{\models}^{\mathcal{M}}_{\alpha} \neg(\varphi \vee \psi)$. By definition 2.4, it holds that $\models^{\mathcal{M}}_{\alpha} (\varphi \vee \psi)$. By what has been proved under (2) it holds that $\models^{\mathcal{M}}_{\alpha} \neg(\neg\varphi \wedge \neg\psi)$ and there fore that $\overline{\models}^{\mathcal{M}}_{\alpha} (\neg\varphi \wedge \neg\psi)$.

(d) Suppose that for a certain world α in the model $\mathcal{M}$ it holds that $\overline{\models}^{\mathcal{M}}_{\alpha} (\neg\varphi \wedge \neg\psi)$. By definition 2.4, it holds that $\models^{\mathcal{M}}_{\alpha} \neg(\neg\varphi \wedge \neg\psi)$. By what has been proved under (2) it holds that $\models^{\mathcal{M}}_{\alpha} (\varphi \vee \psi)$ and there fore that $\overline{\models}^{\mathcal{M}}_{\alpha} \neg(\varphi \vee \psi)$.

q.e.d.

Theorem 5.7

For every sentence of $\mathscr{L}$ of the form exhibited by one of the following schemas holds what follows:

T.	$\models (\Box\varphi \rightarrow \varphi)$
K.	$\models (\Box(\varphi \rightarrow \psi) \rightarrow (\Box\varphi \rightarrow \Box\psi))$
Df$\Diamond$.	$\models (\Diamond\varphi \rightarrow \neg\Box\neg\varphi)$
5.	$\models (\Diamond\varphi \rightarrow \Box\Diamond\varphi)$

Proof

T. Let α a world in a model $\mathcal{M}$. We prove that $\models^{\mathcal{M}}_{\alpha} \Box\varphi \rightarrow \varphi$. According to definition 2.4 this holds if at least one of the following conditions are satisfied: (i) $\overline{\models}^{\mathcal{M}}_{\alpha} \Box\varphi$, (ii) $\models^{\mathcal{M}}_{\alpha} \varphi$, (iii) not $\models^{\mathcal{M}}_{\alpha} \Box\varphi$ and not $\overline{\models}^{\mathcal{M}}_{\alpha} \varphi$. Suppose that all conditions (i)-(iii) are not satisfied. In this case φ cannot be singular, because in such a case condition (iii) would be satisfied. If condition (i) is not satisfied it holds that $\models^{\mathcal{M}}_{\alpha} \Box\varphi$, so that, by theorem 2.1 for every world β in $\mathcal{M}$ it holds that $\models^{\mathcal{M}}_{\beta} \varphi$, so that $\models^{\mathcal{M}}_{\alpha} \varphi$ condition (ii) is satisfied. If condition (ii) is not satisfied then $\overline{\models}^{\mathcal{M}}_{\alpha} \Box\varphi$, so that condition (i) is satisfied. If condition (iii) is not satisfied either $\models^{\mathcal{M}}_{\alpha} \Box\varphi$ or $\overline{\models}^{\mathcal{M}}_{\alpha} \varphi$. However if $\models^{\mathcal{M}}_{\alpha} \Box\varphi$ it holds that $\models^{\mathcal{M}}_{\alpha} \varphi$, so that if $\overline{\models}^{\mathcal{M}}_{\alpha} \varphi$ it holds that $\overline{\models}^{\mathcal{M}}_{\alpha} \Box\varphi$ and condition (i) is satisfied. We conclude that at leat one of the conditions (i)-(iii) is satisfied.

K. Suppose that K. is not valid. Then there exists a world α in a model $\mathcal{M}$ such that not $\models^{\mathcal{M}}_{\alpha} (\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi))$. Since $\to$ is a two-valued truth-function by definition 2.4, in such a case it holds that $\overline{\models}^{\mathcal{M}}_{\alpha} (\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi))$. By definition 2.4 at least one of the two following conditions are satisfied (i) $\models^{\mathcal{M}}_{\alpha} \Box(\varphi \to \psi)$ and not $\models^{\mathcal{M}}_{\alpha} (\Box\varphi \to \Box\psi)$, (ii) not $\overline{\models}^{\mathcal{M}}_{\alpha} \Box(\varphi \to \psi)$ and $\overline{\models}^{\mathcal{M}}_{\alpha} (\Box\varphi \to \Box\psi)$.

Suppose that condition (i) is satisfied. Then (j) for every β in $\mathcal{M}$ it holds that not $\overline{\models}^{\mathcal{M}}_{\beta} (\varphi \to \psi)$ which is equivalent to say that then for every β in $\mathcal{M}$ it holds that $\models^{\mathcal{M}}_{\beta} (\varphi \to \psi)$, and (jj) $\overline{\models}^{\mathcal{M}}_{\alpha} (\Box\varphi \to \Box\psi)$. Consider condition (j). In such a case it holds either that $\overline{\models}^{\mathcal{M}}_{\beta} \varphi$ or $\models^{\mathcal{M}}_{\beta} \psi$ or not $\models^{\mathcal{M}}_{\beta} \varphi$ and not $\overline{\models}^{\mathcal{M}}_{\beta} \psi$. Moreover, under condition (i), it holds that $\overline{\models}^{\mathcal{M}}_{\alpha} (\Box\varphi \to \Box\psi)$ which, by theorem 2.1, entails that at least one of the following conditions are satisfied: (k) $\models^{\mathcal{M}}_{\alpha} \Box\varphi$ and not $\models^{\mathcal{M}}_{\alpha} \Box\psi$, (kk) not $\overline{\models}^{\mathcal{M}}_{\alpha} \Box\varphi$ and $\overline{\models}^{\mathcal{M}}_{\alpha} \Box\psi$. Now if $\overline{\models}^{\mathcal{M}}_{\beta} \varphi$ for every world β, then it holds that $\overline{\models}^{\mathcal{M}}_{\alpha} \Box\varphi$ and rules out both $\models^{\mathcal{M}}_{\alpha} \Box\varphi$ and not $\overline{\models}^{\mathcal{M}}_{\alpha} \Box\varphi$, so that neither condition (k) nor condition (kk) may be satisfied. Suppose that $\models^{\mathcal{M}}_{\beta} \psi$ for every β. This rules out both not $\models^{\mathcal{M}}_{\alpha} \Box\psi$ and $\overline{\models}^{\mathcal{M}}_{\alpha} \Box\psi$. Suppose that for every world β it holds that $\models^{\mathcal{M}}_{\beta} \psi$. This rules out both not $\models^{\mathcal{M}}_{\alpha} \Box\psi$ and $\overline{\models}^{\mathcal{M}}_{\alpha} \Box\psi$. Suppose now that for every world β it holds that not $\models^{\mathcal{M}}_{\beta} \varphi$ and not $\overline{\models}^{\mathcal{M}}_{\beta} \psi$. This condition rules out $\models^{\mathcal{M}}_{\alpha} \Box\varphi$ and also with $\overline{\models}^{\mathcal{M}}_{\alpha} \Box\psi$. So condition (i) cannot be satisfied.

Suppose that condition (ii) is satisfied, so that it holds that not $\overline{\models}^{\mathcal{M}}_{\alpha} \Box(\varphi \to \psi)$ and $\overline{\models}^{\mathcal{M}}_{\alpha} (\Box\varphi \to \Box\psi)$. Condition not $\overline{\models}^{\mathcal{M}}_{\alpha} \Box(\varphi \to \psi)$ amounts to say that for no world β in $\mathcal{M}$ it holds that $\overline{\models}^{\mathcal{M}}_{\beta} (\varphi \to \psi)$.

Condition $\overline{\models}^{\mathcal{M}}_{\alpha}$ $(\Box\varphi \rightarrow \Box\psi)$ requires that at least one of the following conditions are satisfied: (l) $\models^{\mathcal{M}}_{\alpha}$ $\Box\varphi$ and not $\models^{\mathcal{M}}_{\alpha}$ $\Box\psi$, (ll) not $\overline{\models}^{\mathcal{M}}_{\alpha}$ $\Box\varphi$ and $\overline{\models}^{\mathcal{M}}_{\alpha}$ $\Box\psi$. Suppose that not $\overline{\models}^{\mathcal{M}}_{\alpha}$ $\Box(\varphi \rightarrow \psi)$, so that for every world β in $\mathcal{M}$ it holds that not $\overline{\models}^{\mathcal{M}}_{\beta}$ $(\varphi \rightarrow \psi)$, which entails (being $\rightarrow$ a binary connective) that $\models^{\mathcal{M}}_{\beta}$ $(\varphi \rightarrow \psi)$ for every β. By definition 2.4 at least one of the following conditions are then satisfied for every β: (m) $\overline{\models}^{\mathcal{M}}_{\beta}$ φ, (mm) $\models^{\mathcal{M}}_{\beta}$ ψ, (mmm) not $\models^{\mathcal{M}}_{\beta}$ φ and not $\overline{\models}^{\mathcal{M}}_{\beta}$ ψ. Case (m) rules out $\models^{\mathcal{M}}_{\alpha}$ $\Box\varphi$ and therefore also condition (l). Case (m) rules out also that not $\overline{\models}^{\mathcal{M}}_{\alpha}$ $\Box\varphi$ and therefore rules out also condition (ll). Case (mm) rules out that not $\models^{\mathcal{M}}_{\alpha}$ $\Box\psi$ and therefore also condition (l). Case (mm) also rules out that $\overline{\models}^{\mathcal{M}}_{\alpha}$ $\Box\psi$ and therefore rules out also condition (ll). Case (mmm) rules out $\models^{\mathcal{M}}_{\alpha}$ $\Box\varphi$ and therefore rule out also condition (l). Condition (mmm) also rules out that $\overline{\models}^{\mathcal{M}}_{\alpha}$ $\Box\psi$ and therefore rules out condition (ll) as well. This concludes the proof of K.

Df$\Diamond$. By theorem 2.1 and theorem 5.2 $\models$ $(\neg\Box\neg\varphi \leftrightarrow \neg\neg\Diamond\neg\neg\varphi)$. Applying theorem 5.2 two times and replacement equivalents with equivalents to the preceding formula we get $\models$ $(\neg\Box\neg\varphi \leftrightarrow \Diamond\varphi)$.

5. Let α be any world in a model $\mathcal{M}$. We prove that $\models^{\mathcal{M}}_{\alpha}$ $(\Diamond\varphi \rightarrow \Box\Diamond\varphi)$. According to definition 2.4, this holds if at least one of the following conditions are satisfied: (i) $\overline{\models}^{\mathcal{M}}_{\alpha}$ $\Diamond\varphi$, (ii) $\models^{\mathcal{M}}_{\alpha}$ $\Box\Diamond\varphi$, (iii) not $\models^{\mathcal{M}}_{\alpha}$ $\Diamond\varphi$ and not $\overline{\models}^{\mathcal{M}}_{\alpha}$ $\Box\Diamond\varphi$. Suppose that no condition (i)-(iii) is satisfied, so that their respective negations are *all* satisfied: (j) not $\overline{\models}^{\mathcal{M}}_{\alpha}$ $\Diamond\varphi$, (jj) not $\models^{\mathcal{M}}_{\alpha}$ $\Box\Diamond\varphi$, (jjj) either $\models^{\mathcal{M}}_{\alpha}$ $\Diamond\varphi$ or $\overline{\models}^{\mathcal{M}}_{\alpha}$ $\Box\Diamond\varphi$. Condition (j) may be reformulated by saying that either (j1) there exists a world β in $\mathcal{M}$ such that $\models^{\mathcal{M}}_{\beta}$ φ or (j2) no world γ in $\mathcal{M}$ is such that $\overline{\models}^{\mathcal{M}}_{\gamma}$ φ. Condition (jj) may be reformulated by saying that either (jj1) there exists a world β in $\mathcal{M}$ such that $\overline{\models}^{\mathcal{M}}_{\beta}$ $\Diamond\varphi$ or (jj2) no world γ in $\mathcal{M}$ is such that $\models^{\mathcal{M}}_{\gamma}$ $\Diamond\varphi$. Condition (jjj) may be reformulate by saying that either (jjj1) there exists a world β in $\mathcal{M}$ such that $\models^{\mathcal{M}}_{\beta}$ φ or there exists a world γ in $\mathcal{M}$ such that $\overline{\models}^{\mathcal{M}}_{\beta}$ $\Diamond\varphi$. Suppose that (j1) is satisfied. Clearly (j1) is incompatible with condition (jj1) which entails that for no β in $\mathcal{M}$ it holds that $\models^{\mathcal{M}}_{\beta}$ φ. So necessarily, if (j1) is satisfied, (jj2) must be satisfied too, so that no γ in $\mathcal{M}$ is such that $\models^{\mathcal{M}}_{\gamma}$ $\Diamond\varphi$. But (jj2) may be reformulate by saying that no γ in M is such that $\models^{\mathcal{M}}_{\gamma}$ φ, which again is at odds with (j1). So, let us consider (j2). Clearly, (j2) is incompatible with (jj1) which entails that there exists β in $\mathcal{M}$ such that $\overline{\models}^{\mathcal{M}}_{\beta}$ φ. So, if (j2) is satisfied, it must

be satisfied also (jj2). Now, (jj2) may be satisfied only if $\models_{\mathcal{M}} \varphi \leftrightarrow \natural$. However, this is incompatible with (jjj), which entails that there exists β in $\mathcal{M}$ such that $\models^{\mathcal{M}}_{\beta} \varphi$, while for every β in $\mathcal{M}$ it holds that not $\models^{\mathcal{M}}_{\beta} \natural$. So conditions (j), (jj) and (jjj) cannot be simultaneously satisfied. This concludes the proof of 5.

q.e.d.

Theorem 5.7 proves that our semantics encompasses S5 logic. In fact, tautologous schemas are valid on the set of ordinary sentences (which is closed with respect to the connectives $\Diamond, \Box, \wedge, \vee, \neg$), so that valid ordinary sentences may be regarded as S5-valid sentences.

Theorem 5.8

For every sentence φ of $\mathscr{L}$ it holds that $\models \varphi$ iff $\models \Box{\uparrow}\varphi$.

Proof

Suppose $\models \varphi$. Then by definition 2.13 φ is true at every world in all models. By definition 2.4 also ${\uparrow}\varphi$ is true at every world in all models. By theorem 2.1, also $\Box{\uparrow}\varphi$ is true at every world in all models, so that $\models \Box{\uparrow}\varphi$.

Suppose that $\models \Box{\uparrow}\varphi$. By Theorem 2.1 ${\uparrow}\varphi$ is true at every world in all models and by definition 2.13, φ is true at every world in all models, so that $\models \varphi$.

q.e.d.

Definition 5.3 (Quasi-atomic sentences)

A sentence φ is said to be quasi-atomic iff it is either of the form ${\uparrow}\varphi$ or of the form ${\uparrow}\neg\varphi$, where φ is either an atomic wff or $\varphi \approx \bot$.

Definition 5.4 (Normal quasi-classical form)

A sentence φ is said to be in normal quasi-classical (nqc) form iff the following conditions are satisfied:

(i) every occurrence of a subformula of the form ${\uparrow}\psi$, if any, is such that ψ is either an atomic sentence or the negation of an atomic sentence or a quasi-atomic sentence;

(ii) φ does not contain any occurrence of the conditioning symbol '|';

(iii) every occurrence of atomic sentences is immediately preceded either by the symbol '$\uparrow$' or by the sequence of symbols '${\uparrow}\neg$'.

Theorem 5.9

Every sentence in normal quasi-classical form is an ordinary sentence.

Proof

Omitted.

q.e.d.

Theorem 5.10

For every sentence φ both ${\uparrow}\varphi$ and ${\uparrow}\neg\varphi$ are logically equivalent to a sentence ψ in normal quasi-classical form.

Proof

By induction on the construction of φ. Base. If $\varphi \approx \bot$, then both $\uparrow\varphi$ and $\uparrow\neg\varphi$ are, by definition 5.3 quasi-atomic sentences, so that φ is, by definition 5.4 in nqc form. If φ is an atomic sentence then both $\uparrow\varphi$ and $\uparrow\neg\varphi$ are, by definition 5.4 in nqc form.

Step.

(a) $\varphi \approx \neg\psi$, where $\uparrow\psi$ and $\uparrow\neg\psi$ are logically equivalent, respectively, to sentences χ and ζ in nqc form.

  (i) $\uparrow\varphi$. $\uparrow\varphi \approx \uparrow\neg\psi$. But $\uparrow\neg\psi$ is assumed to be logically equivalent to ζ which is, by inductive hypothesis, in nqc form.

  (ii) $\uparrow\neg\psi$. $\uparrow\varphi \approx \uparrow\neg\neg\psi$, where $\uparrow\neg\neg\psi$ is logically equivalent, by theorem 5.2 to $\uparrow\psi$ which, in turn, is logically equivalent to χ, which is, by inductive hypothesis, in nqc form.

(b) $\varphi \approx \uparrow\psi$, where $\uparrow\psi$ and $\uparrow\neg\psi$ are logically equivalent, respectively, to sentences χ and ζ in nqc form.

  (i) $\uparrow\varphi \approx \uparrow\uparrow\psi$ which is logically equivalent to $\uparrow\psi$. Now, $\uparrow\psi$ is assumed to be logically equivalent to χ which is assumed to be, by inductive hypothesis, in nqc form.

  (ii) $\uparrow\neg\varphi \approx \uparrow\neg\uparrow\psi$ and since $\uparrow\psi$ is logically equivalent to a sentence χ in nqc form, by (a) $\neg\chi$ is also in nqc form, so that $\uparrow\neg\chi$ too, as proved above, is in nqc form.

(c) $\varphi \approx (\psi \wedge \chi)$ where $\uparrow\psi$, $\uparrow\chi$, $\uparrow\neg\psi$, and $\uparrow\neg\chi$ are equivalent, respectively, to sentences ψ_1, χ_1, ψ_2, and χ_2 in nqc form.

  (i) $\uparrow\varphi \approx \uparrow(\psi \wedge \chi)$, which, by theorem 5.5 is logically equivalent to $(\uparrow\varphi \wedge \uparrow\psi)$ that, in turn, is logically equivalent to $(\psi_1 \wedge \chi_1)$, which is, by inductive hypothesis, in nqc form.

  (ii) $\uparrow\neg(\psi \wedge \chi)$ is, by theorem 5.6, logically equivalent to $\uparrow(\neg\psi \vee \neg\chi)$, which is, in turn, logically equivalent to $((\uparrow\psi \vee \uparrow\chi) \vee (\neg\Diamond(\uparrow\neg\psi \wedge \uparrow\neg\chi) \wedge (\Diamond\uparrow\psi \vee \Diamond\uparrow\chi)))$ which is logically equivalent to $((\psi_1 \vee \chi_1) \vee (\neg\Diamond(\psi_2 \wedge \chi_2) \wedge (\Diamond\psi_1 \vee \Diamond\chi_1)))$ which is, in the presence of the inductive hypothesis, logically equivalent to a sentence in nqc form.

(d) $\varphi \approx \Diamond\psi$, where $\uparrow\psi$ is logically equivalent to a sentence χ in nqc form.

  (i) By theorem 5.5, $\uparrow\varphi \approx \uparrow\Diamond\psi$ is logically equivalent to $\Diamond\uparrow\psi \approx \Diamond\chi$. Now, if χ is in nqc form also $\Diamond\chi$ and $\Diamond\uparrow\psi \approx \Diamond\chi$ are in nqc form in the presence of the inductive hypothesis.

  (ii) $\uparrow\neg\varphi \approx \uparrow\neg\Diamond\psi$, which is logically equivalent, by theorem 5.5, to $(\uparrow(\psi \vee \neg\psi) \wedge \neg\Diamond\uparrow\psi)$. Now, $\uparrow(\psi \vee \neg\psi)$ is logically equivalent, by theorem 5.5 to $((\uparrow\psi \vee \uparrow\neg\psi) \vee (\neg\Diamond\uparrow\neg\psi \wedge \Diamond\uparrow\psi))$, which, in virtue of the inductive hypothesis, is logical equivalent to a nqc sentence.

(e) $\varphi \approx (\psi \vee \chi)$, where $\uparrow\psi$ and $\uparrow\chi$, are logically equivalent, respectively, to sentences ψ_1, χ_1, ψ_2, and χ_2 in nqc form.

  (i) By theorem 5.5, $\uparrow(\psi \vee \chi)$ is logically equivalent to $((\uparrow\psi \vee \chi_1) \vee (\neg\Diamond(\psi_2 \wedge \chi_2) \wedge (\Diamond\psi_1 \vee \Diamond\chi_1)))$ which is, by the inductive hypothesis, logically equivalent to a sentence in nqc form.

  (ii) By theorem 5.6, $\uparrow\neg(\psi \vee \chi)$ is, logically equivalent to $\uparrow(\neg\psi \wedge \neg\chi)$. Now, $\uparrow(\neg\psi \wedge \neg\chi)$ is logically equivalent, by theorem 5.5 to $(\uparrow\neg\psi \wedge \uparrow\neg\chi)$ which is, in turn, logically equivalent to $(\psi_2 \wedge \chi_2)$, which is, by the inductive hypothesis, in nqc form.

(f) $\varphi \approx (\psi \to \chi)$ where $\uparrow\psi$, $\uparrow\chi$, $\uparrow\neg\psi$, $\uparrow\neg\chi$ are logically equivalent, respectively, to ψ_1, χ_1, ψ_2, χ_2 in nqc form.

  (i) By theorem 5.5, $\uparrow(\psi \to \chi)$ is logically equivalent to $((\uparrow\chi \vee (\neg\uparrow\psi \wedge \neg\uparrow\neg\chi)) \vee \uparrow\neg\psi)$ which is logically equivalent to $(\chi_1 \vee (\neg\psi_1 \wedge \neg\chi_2) \vee \psi_2)$ which is, by the inductive hypothesis, in nqc form.

  (ii) $\uparrow\neg(\psi \to \chi)$ is logically equivalent to $((\psi_1 \wedge \neg\chi_1) \vee (\chi_2 \wedge \neg\psi_2))$ which is, by inductive hypothesis, in nqc form.

(g) $\varphi \approx (\chi \mid \psi)$, where $\uparrow\psi$, $\uparrow\chi$, $\uparrow\neg\psi$, $\uparrow\neg\chi$ are logically equivalent, respectively, to ψ_1, χ_1, ψ_2, χ_2 in nqc form.

  (i) By theorem 5.5, $\uparrow\varphi$ is logically equivalent to $((\uparrow\psi \wedge \uparrow\chi) \vee (\Diamond(\uparrow\psi \wedge \uparrow\chi) \wedge \neg\Diamond(\uparrow\psi \wedge \uparrow\neg\chi)))$ which, in turn, is logically equivalent to $((\psi_1 \wedge \chi_1) \vee (\Diamond(\psi_1 \wedge \chi_1) \wedge \neg\Diamond(\psi_1 \wedge \chi_2))$, which is, by inductive hypothesis, in nqc form.

  (ii) $\uparrow\neg\varphi$ is logically equivalent to $\uparrow(\chi_2 \mid \psi)$, and it follows from what has been proved above that it is logically equivalent to a sentence in nqc form.

q.e.d.

At this point we will introduce two sub-languages, named respectively $\mathscr{L}_1$ and $\mathscr{L}_2$. The language $\mathscr{L}_1$ has all the symbols of $\mathscr{L}$ except the conditioning symbol '|'. Moreover, the atomic sentences of $\mathscr{L}_1$ are the quasi-atomic sentences of $\mathscr{L}$. All the sentences of $\mathscr{L}_1$ are ordinary sentences and, from a semantical point of view, essentially two-valued (that is two-valued in every model). The language $\mathscr{L}_2$ is just a classic modal sentential language. It lacks, of course, the symbol '|'.

Definition 5.5 (The sub-language $\mathscr{L}_1$)

Let QA the set of all the quasi-atomic sentences of $\mathscr{L}$. The class of sentences of $\mathscr{L}_1$ is recursively defined as follows:

1. $\perp$ is a sentence of $\mathscr{L}_1$.
2. Every quasi-atomic sentence of $\mathscr{L}$ is a sentence of $\mathscr{L}_1$.
3. If φ is a sentence of $\mathscr{L}_1$ then also $\neg\varphi$ and $\Diamond\varphi$ are sentences of $\mathscr{L}_1$.

4. If φ and ψ are sentences of $\mathscr{L}_1$ then also $(\varphi \vee \psi)$, $(\varphi \wedge \psi)$, $(\varphi \rightarrow \psi)$ are sentences of $\mathscr{L}_1$.

5. Nothing else is a sentence of $\mathscr{L}_1$.

Theorem 5.11

Every sentence in normal quasi-classical form is a sentence of $\mathscr{L}_1$.

Proof
Omitted.

q.e.d.

Theorem 5.12

Every sentence in $\mathscr{L}_1$ is an ordinary sentence.

Proof
Omitted.

q.e.d.

Definition 5.6 (The sub-language $\mathscr{L}_2$)

The class of sentences of $\mathscr{L}_2$ is recursively defined as follows:

1. $\perp$ is a sentence of $\mathscr{L}_2$.

2. Every atomic sentence of $\mathscr{L}$ is a sentence of $\mathscr{L}_2$.

3. If φ is a sentence of $\mathscr{L}_2$ then also $\neg\varphi$ and $\Diamond\varphi$ are sentences of $\mathscr{L}_2$.

4. If φ and ψ are sentences of $\mathscr{L}_2$ then also $(\varphi \vee \psi)$, $(\varphi \wedge \psi)$, $(\varphi \rightarrow \psi)$ are sentences of $\mathscr{L}_2$

5. Nothing else is a sentence of $\mathscr{L}_2$

Theorem 5.13

Let $\mathfrak{M}$ the class of models of $\mathscr{L}$ such that for every atomic sentence $\mathbb{P}_n$ it holds that $P_n = W$. Then a sentence φ of $\mathscr{L}_2$ is valid in every model in $\mathfrak{M}$ iff it is a theorem of the modal system S5.

Proof
Omitted.

q.e.d.

Theorem 5.14

Every sentence φ of $\mathscr{L}$ is logically equivalent to a syntactically simple tri-event.

Proof
By Theorem 5.5, φ is logically equivalent to $(\uparrow\varphi \mid \updownarrow\varphi)$. By Theorem 5.10 we can assume, without loss of generality, that $\uparrow\varphi$ and $\updownarrow\varphi$ are in nqc form. From this follows, by definition 5.1 that $(\uparrow\varphi \mid \updownarrow\varphi)$ is a syntactically simple tri-event.

q.e.d.

Theorem 5.15

For every sentence φ of $\mathscr{L}$, if $\models \varphi$ then there is a sentence ψ of $\mathscr{L}_2$, effectively associated to φ, which, by definition 5.6, is valid in those models of $\mathscr{L}$ in which for every atomic sentence $\mathbb{P}_n$ occurring in φ it holds that $P_n = W$.

Proof

Suppose $\models \varphi$. By Theorem 5.8 we may assume, without loss of generality that $\varphi = \uparrow\Box\varphi$. By Theorem 5.10 φ is logically equivalent to a sentence χ in nqc form. Now let $\mathbb{P}_1 \ldots \mathbb{P}_k$ the distinct atomic sentences occurring in χ. Let us associate with every $\mathbb{P}_i (1 \leq i \leq k)$ two atomic sentences $\mathbb{P}_i^1$ and $\mathbb{P}_i^2$ not belonging to the set $\{\mathbb{P}_1, \ldots, \mathbb{P}_k\}$. Let ψ' be obtained by replacing, for every $i(1 \leq i \leq k)$ in χ every occurrence of $\uparrow\mathbb{P}_i\}$ by $\mathbb{P}_i^1$ and every occurrence of $\uparrow\mathbb{P}_i$ by $\mathbb{P}_i^2$. Let ψ'' be the sentence $(\neg\Diamond(\mathbb{P}_1^1 \wedge \mathbb{P}_1^2) \wedge \ldots \wedge \neg\Diamond(\mathbb{P}_k^1 \wedge \mathbb{P}_k^2))$. Let ψ be the sentence $(\psi'' \rightarrow \psi')$. ψ is a sentence of $\mathscr{L}_2$. We prove that $\models_{\mathcal{M}} \psi$ for every model $\mathcal{M}$ such that for every atomic sentence $\mathbb{P}_n$ occurring in φ it holds that $P_n = W$. Under the hypothesis that $\models \varphi$, it holds that $\models_{\mathcal{M}} \varphi$. Given two models $\mathcal{M}_1$ and $\mathcal{M}_2$, let us consider the following relation $\mathcal{M}_1\ R_\varphi\ \mathcal{M}_2$ between $\mathcal{M}_1$ and $\mathcal{M}_2$ that holds iff (1) $\mathcal{M}_1$ and $\mathcal{M}_2$ share the same set of worlds W, (2) for every world $\alpha \in W$, and every quasi-atomic sub-sentence ζ of $\models_\alpha^{\mathcal{M}_1} \zeta$ iff $\models_\alpha^{\mathcal{M}_2} \zeta$, (3) for every world $\alpha \in W$ and every quasi-atomic sub-sentence ζ of φ, $\overline{\models}_\alpha^{\mathcal{M}_1} \psi$ iff $\overline{\models}_\alpha^{\mathcal{M}_2} \psi$. R_φ is an equivalence relation, according to which the set of models are partitioned in equivalences classes. Now, every quasi-atomic sentence of the form $\uparrow\mathbb{P}_i$ occurring in φ is true at a given world α (in any model $\mathcal{M}$) iff the quasi-atomic sentence $\uparrow\neg\mathbb{P}_i$ is false at α, and it is false at α iff $\uparrow\neg\mathbb{P}_i$ is true at α. So there is a one-to-one correspondence between the set of the equivalence classes and the set of those models $\mathcal{M}$ such that $P_n = W$ and for every world α in $\mathcal{M}$ it holds that $\models_\alpha^{\mathcal{M}} \mathbb{P}_i^1$ iff $\models_\alpha^{\mathcal{M}} \uparrow\mathbb{P}_i$ and $\overline{\models}_\alpha^{\mathcal{M}} \mathbb{P}_i^2$ iff $\overline{\models}_\alpha^{\mathcal{M}} \uparrow\neg\mathbb{P}_i$. We may associate to each equivalence class a single model satisfying that property. φ and ψ have the same truth-value at each world in the corresponding models. Since $\models \varphi$ also $\models_\alpha^{\mathcal{M}} \psi'$ for every world α in such models. Let $\mathcal{M}'$ be a residual model in which for some world and some i, both $\mathbb{P}_i^1$ and $\mathbb{P}_i^2$ are true. In such a case ψ'' is false and therefore ψ is true. So ψ is true at every world in every model such that $P_n = W$.

q.e.d.

Corollary 5.1

For every sentence φ of $\mathscr{L}$, if $\models \varphi$ then there is a sentence ψ of $\mathscr{L}_2$, effectively associated to φ, which is a theorem of the modal system S5.

Proof

Immediate from theorem 5.15, theorem 5.13 and the completeness theorem for S5.

q.e.d.

Corollary 5.2

There is a decision procedure for satisfiability in every model, validity in every model and logical consequence among the sentences of $\mathscr{L}$.

Proof

Immediate from corollary 5.1, and the fact that S5 logic is decidable.

q.e.d.

6. Probability

6.1 Probability of Tri-events in General

In the original de Finetti's theory of probability, the probability is first defined for ordinary events and as a second step extended to any tri-event by the equation $\frac{\mathbf{P}(\uparrow\varphi)}{\mathbf{P}(\updownarrow\varphi)}$ provided $\mathbf{P}(\updownarrow\varphi) > 0$. The axioms of probability for two-valued sentences are the standard axioms for finite probability. This definition applies unmodified also to our modal theory, even if the truth conditions of φ (and therefore of $\uparrow\varphi$ and $\updownarrow\varphi$) are different. Quasi-tautologies (that are converted into valid sentences in all models of our theory) have probability 1 in de Finetti's theory as in our theory. Dually, quasi-contradictions (that are converted in countervalid sentences in all models in our theory) have probability 0 in de Finetti's theory as in our theory. Modal sentences — that is sentences of the form $\Diamond\varphi$ or $\Box\varphi$ — that are absent in de Finetti's theory, have either probability 1 or probability 0.

An equivalent alternative approach is to provide a set of axioms over all tri-events that is a generalisation of the standard theory. Several explanations of Lewis' triviality results assume that the probability of conditionals obeys the standard laws of probability. As already observed, this assumption is natural if we consider conditional sentences as two-valued sentences obeying the standard logic but is no longer reasonable if we represent conditionals as tri-events. The standard axioms of probability apply to Boolean algebras of sentences (up to logical equivalence), while the corresponding algebraic structure for tri-events is *not* a Boolean algebra. What we need is a more general theory of probability such that naturally comes down to the standard probability concerning ordinary sentences.

Since our theory is throughout a modal theory, and the modal notion involved in probability theory are model-relative, we propose a set of axioms regarding a given model $\mathcal{M}$. The model-relative probability function will be denoted $\mathbf{P}^{\mathcal{M}}$. Probability is defined for every sentence of $\mathscr{L}$ except for singular sentences. For every sentence φ of $\mathscr{L}$, the probability $\mathbf{P}^{\mathcal{M}}(\varphi)$ is considered *undefined*.[11] Accordingly, the values of the metalinguistic variables φ and ψ below are assumed to be such that for every α in $\mathcal{M}$ it holds that $\models^{\mathcal{M}}_{\alpha} \Diamond\updownarrow\varphi$ and $\models^{\mathcal{M}}_{\alpha} \Diamond\updownarrow\psi$, that is we assume that neither φ nor ψ are singular in $\mathcal{M}$.

A1 $\mathbf{P}^{\mathcal{M}}(\varphi) \geq 0$

A2 $\mathbf{P}^{\mathcal{M}}(\bot) = 0$

A3 $\mathbf{P}^{\mathcal{M}}(\uparrow\varphi) = \mathbf{P}^{\mathcal{M}}(\varphi) \times \mathbf{P}^{\mathcal{M}}(\updownarrow\varphi)$

A4 If ψ is an $\mathcal{M}$-consequence of φ, $\mathbf{P}^{\mathcal{M}}(\varphi) \leq \mathbf{P}^{\mathcal{M}}(\psi)$

[11] We explain in the next section why the closure move of assigning probability 1 to every singular sentence is not appropriate in the current approach.

A5 $\mathbf{P}^{\mathcal{M}}(\neg\varphi) = 1 - \mathbf{P}^{\mathcal{M}}(\varphi)$

Theorem 6.1

Let $\mathcal{M}$ be a model. Let $\varphi = (\psi \mid \chi)$ where ψ and χ express two-valued propositions in $\mathcal{M}$. Let $\mathbf{P}^{\mathcal{M}}$ a probability function over $\mathcal{M}$ such that $\mathbf{P}^{\mathcal{M}}(\chi) > 0$. Then it holds that

$$\mathbf{P}^{\mathcal{M}}(\varphi) = \frac{\mathbf{P}^{\mathcal{M}}(\psi \wedge \chi)}{\mathbf{P}^{\mathcal{M}}(\chi)}$$

Proof

By definition 2.4 $\models^{\mathcal{M}}_{\alpha} \uparrow\varphi$ iff $\models^{\mathcal{M}}_{\alpha} \psi$ and $\models^{\mathcal{M}}_{\alpha} \chi$, that is iff $\models^{\mathcal{M}}_{\alpha} \uparrow\psi$ and $\models^{\mathcal{M}}_{\alpha} \uparrow\chi$. Now, $\uparrow\psi$ and $\uparrow\chi$ are, respectively, logically $\mathcal{M}$–equivalent to ψ and χ, since they express two-valued propositions. By Theorem 5.5 $\uparrow(\psi \wedge \chi)$ is logically equivalent to $(\uparrow\psi\uparrow\chi)$. Since $(\uparrow\psi \wedge \uparrow\chi)$ is $\mathcal{M}$–equivalent to $(\psi \wedge \chi)$, $\uparrow\varphi$ is $\mathcal{M}$–equivalent to $(\psi \wedge \chi)$. Consider $\updownarrow\varphi$. By theorem 2.1 $\updownarrow\varphi$ iff either $\models^{\mathcal{M}}_{\alpha} \varphi$ or $\overline{\models}^{\mathcal{M}}_{\alpha} \varphi$. In both cases $\models^{\mathcal{M}}_{\alpha} \updownarrow\varphi$ iff $\models^{\mathcal{M}}_{\alpha} \chi$, so that $\updownarrow\varphi$ is logically equivalent to χ. By axiom A3, it holds that

$$\mathbf{P}^{\mathcal{M}}(\varphi) = \frac{\mathbf{P}^{\mathcal{M}}(\uparrow\varphi)}{\mathbf{P}^{\mathcal{M}}(\updownarrow\varphi)} = \frac{\mathbf{P}^{\mathcal{M}}(\psi \wedge \chi)}{\mathbf{P}^{\mathcal{M}}(\chi)}$$

q.e.d.

6.2 *p*–Entailment and *p*-Consistency

Adams oscillates between two notions of *p*-consistency. In his 1975 book, his definition was based on the idea that the probability of a conditional is *proper* only if the antecedent has a probability greater than 0. The definition is extended to a set X of formulas so that an assignment is proper iff it is proper to every conditional formula of X (Adams 1975: 49-50). *p*-consistency and *p*-inconsistency are defined considering only proper assignments. More exactly, X is *p*-consistent iff for all real numbers $\epsilon > 0$ there exist probability assignments p which are proper for X such that $\mathbf{P}(\varphi) \geq 1 - \epsilon$ for every element φ of X.

As a result, a set of conditionals of the form $\{\varphi \Rightarrow \psi, \varphi \Rightarrow \neg\psi\}$ is considered as *p*-inconsistent. Later, Adams changed his mind and returning to his previous view (Adams 1965, 1966), got rid of the notion of proper assignment and assumed again that when $\mathbf{P}(\varphi) = 0$, $\mathbf{P}(\psi \mid \varphi) = 1$. In such a case, the set $\{\varphi \Rightarrow \psi, \varphi \Rightarrow \neg\psi\}$ is considered as *p*-consistent. However, Adams remarked that

> [P]robabilistic consistency is considerably more difficult to characterize, especially because it has more than one sense. In one sense A⇒B and A⇒ ∼B are consistent, since they are both certain when p(A) = 0, and therefore p(A⇒B) = p(A⇒ ∼B) = 1. However, there seems to be another sense, for instance in which 'If Anne goes to the party then Ben will go, and if she goes to the party then Ben won't go' would be

'absurd' even though p(A⇒B) and p(A⇒ ~B) can both equal 1, and it is important to make this sense clear (Adams 1998: 181).

McGee (1994) adopted the so-called Popper-functions to allow conditional probability $\mathbf{P}(\psi \mid \varphi)$ to be defined even when φ is impossible. According to Popper functions $\mathbf{P}(\varphi \mid \bot) = 1$ for every sentence φ, but may have any value if φ is a consistent sentence such that $\mathbf{P}(\varphi) = 0$. In our context, we *must* drop the assumption that $\mathbf{P}(\varphi \mid \bot) = 1$, because it would imply that $\mathbf{P}(\natural) = \mathbf{P}(\top)$, so that two logically nonequivalent sentences would have the same probability value for every probability function.[12] Moreover, if the probability of an indicative conditional is the conditional expectation of a truth-value, clearly it is undefined when the antecedent is impossibly true. One is tempted to considered undefined the probability for every tri-event φ that, in a given model $\mathcal{M}$, is such that $\mathbf{P}(\updownarrow\varphi) = 0$. Indeed, if there is a reasonable certainty that a tri-event has no truth value, it is natural to assume that it has no probability value too. $\mathbf{P}(\updownarrow\varphi) = 0$ is always true when in a conditional sentence of the form $(\psi \mid \varphi)$, it holds that $\mathbf{P}(\varphi) = 0$. o, this assumption corresponds to the standard assumption that conditional probability is undefined when the conditioning event has probability 0. However, our axioms for probability do not entail this. So, we prefer to stick with Adams 1975 approach, defining *p*-entailment and *p*-consistency in terms of those probability assignments that assign a probability greater than 0 to every sentence of the form $\updownarrow\varphi$ for every involved sentence φ, while allowing probability functions to be defined even when the antecedent has probability 0 (short of being countervalid).

In the light of these considerations, we may define *p*-entailment and *p*-consistency for tri-events as follows.

Definition 6.1 (*p*-entailment in a model $\mathcal{M}$)

Let $X \cup \{\varphi\}$ be a set of sentences of $\mathscr{L}$. Let $\mathcal{M}$ be a model. X probabilistically entails φ in $\mathcal{M}$ iff either

(A) for all $\epsilon > 0$ there exists $\delta > 0$ such that for all coherent probability assignments $\mathbf{P}^{\mathcal{M}}$ for $\mathscr{L}$ such that for every element ψ belonging to X such that $\mathbf{P}^{\mathcal{M}}(\updownarrow \psi) > 0$, it holds that if $\mathbf{P}^{\mathcal{M}}(\chi) \geq 1 - \delta$ for all $\chi \in X$, then $\mathbf{P}^{\mathcal{M}}(\varphi) \geq 1 - \epsilon$
or

(B) $X \cup \{\varphi\}$ is singular.

Condition (b) is not present in the original Adams definition because singular sentences are not present in the language that he considered, and we have added clause (b) for closure reasons.

[12] This happens also in Adams' probability logic (1965, 1966, 1998), while does not happen in McGee's probability logic (1994).

Definition 6.2 (p-consistency in a model $\mathcal{M}$)

X be a finite set of sentences of $\mathscr{L}$. Let $\mathcal{M}$ be a model. X is probabilistically consistent iff for all real numbers $\epsilon > 0$, there exists at least a probability assignment such that for every element φ of X it holds that $\mathbf{P}^{\mathcal{M}}(\updownarrow \varphi) > 0$ and $\mathbf{P}^{\mathcal{M}}(\varphi) \geq 1 - \epsilon$.

6.3 Satisfiability and Adams' Confirmability

We may compare our satisfiability notion to Adams' *confirmability*. We prove the link between the two notions by the next theorem. Confirmability, in turn, may be easily defined concerning a given model. Let Γ be a finite set of syntactically simple sentences of the form $(\psi \mid \varphi)$ where φ is not countervalid. We may assume, without loss of generality, that φ and ψ are in nqc form. Let Γ' the set of the sentences in nqc form occurring in at least a sentence of Γ. Let $\mathcal{M}$ be a model. Every element of Γ' is either true or false at every world in $\mathcal{M}$. For every world α in $\mathcal{M}$, we define an assignment $t_\alpha^{\mathcal{M}}$ of truth-values to the elements of Γ' in the following way: $t_\alpha^{\mathcal{M}}(\varphi) = \text{true}$ iff $\models_\alpha^{\mathcal{M}} \varphi$ and $t_\alpha^{\mathcal{M}}(\varphi) = \text{false}$ iff $\overline{\models}_\alpha^{\mathcal{M}} \varphi$. We say that an element $(\psi \mid \varphi)$ of Γ is *verified* under the truth-assignment $t_\alpha^{\mathcal{M}}$ iff $\models_\alpha^{\mathcal{M}} \varphi$ and $\models_\alpha^{\mathcal{M}} \psi$, that $(\psi \mid \varphi)$ is *falsified* under the truth-assignment $t_\alpha^{\mathcal{M}}$ iff $\models_\alpha^{\mathcal{M}} \varphi$ and $\overline{\models}_\alpha^{\mathcal{M}} \psi$, and that $(\psi \mid \varphi)$ is neither verified nor falsified iff $\overline{\models}_\alpha^{\mathcal{M}} \varphi$. Notice that this notion of verification does not coincide with truth in the model $\mathcal{M}$. For example, if $\models_{\mathcal{M}} \psi$ and φ is false at a given world α, $t_\alpha^{\mathcal{M}}(\psi \mid \varphi)$ is neither verified nor falsified, while being true at every world, according to our semantics. We say that Γ is *confirmed* by $t_\alpha^{\mathcal{M}}$ iff, under the truth-assignment $t_\alpha^{\mathcal{M}}$, no member of Γ is falsified and at least one is verified. We say that Γ is *disconfirmed* by $t_\alpha^{\mathcal{M}}$ iff, under the truth-assignment $t_\alpha^{\mathcal{M}}$, no member of Γ is verified and at least one is falsified. We say that Γ is *confirmable* at the model $\mathcal{M}$ iff there exists a truth-assignment $t_\alpha^{\mathcal{M}}$ that confirms it and is *disconfirmable* at the model $\mathcal{M}$ iff there exists a truth-assignment that disconfirms it.

Theorem 6.2

Let Γ be a finite set of syntactically simple tri-events of the form $(\psi \mid \varphi)$ where φ and ψ are in nqc form and φ is not countervalid in $\mathcal{M}$. Γ is satisfiable at a model $\mathcal{M}$ iff every non-empty subset of Γ is confirmable at $\mathcal{M}$.

Proof

We prove first (a) that if Γ is satisfiable at a model $\mathcal{M}$, then every non-empty subset of Γ is confirmable at $\mathcal{M}$, and (b) that if every non-empty subset of Γ is confirmable at $\mathcal{M}$, then Γ is satisfiable at $\mathcal{M}$.

(a) Suppose that Γ is satisfiable at a model $\mathcal{M}$. If Γ is empty, then it is trivially confirmable. No element of Γ is singular by hypothesis. By definition 2.7, for every nonempty subset Γ' of Γ there is a world α in $\mathcal{M}$ such that for every element φ in Γ' it holds that not $\overline{\models}_\alpha^{\mathcal{M}} \varphi$ and there is an element ψ in Γ' such that $\models_\alpha^{\mathcal{M}} \psi$. Suppose $\psi = (\xi \mid \chi)$. There are two cases: (i) $\models_\alpha^{\mathcal{M}} \chi$ and $\models_\alpha^{\mathcal{M}} \xi$, and (ii) $\models_\alpha^{\mathcal{M}} \psi$ but either not $\models_\alpha^{\mathcal{M}} \xi$

or not $\models_{\alpha}^{\mathcal{M}} \chi$. In the case (i) Γ' is confirmed by the truth-assignment $t_{\alpha}^{\mathcal{M}}$, in the case (ii) ψ is a near-tautology, so that at every world α in $\mathcal{M}$ it holds either it is verified or falsified by $t_{\alpha}^{\mathcal{M}}$. Suppose *ab absurdo* that for every world β at which both $\models_{\beta}^{\mathcal{M}} \chi$ and $\models_{\beta}^{\mathcal{M}} \xi$ hold, there is an element ζ of Γ' such that it holds $\overline{\models}_{\beta}^{\mathcal{M}} \beta$. If ζ is a near-contradiction, then ζ is countervalid because we assumed that it is not singular. In this case, it is false at every world, which contradicts the hypothesis that Γ is satisfiable. If ζ is not a near-tautology then there is a world γ at which it holds that $\models_{\gamma}^{\mathcal{M}} \zeta$ is true while not $\overline{\models}_{\gamma}^{\mathcal{M}} \psi$. This result holds for every other element of Γ'. However, if so Γ' is confirmed by at least one truth-assignment, and therefore is confirmable.

(b) Suppose that every non-empty subset Γ' of Γ is confirmable at $\mathcal{M}$. This means that there exists a world α in $\mathcal{M}$ such that under the truth-assignment $t_{\alpha}^{\mathcal{M}}$, no element of Γ' is falsified and at least one of them is verified. Now every not falsified element φ is such that not $\overline{\models}_{\alpha}^{\mathcal{M}} \varphi$ and every verified element is such that $\models_{\alpha}^{\mathcal{M}} \varphi$. It follows that Γ is satisfiable at $\mathcal{M}$.

q.e.d.

In the light of theorem 5.14, the assumptions that the elements of Γ are simple and that φ and ψ are in nqc form implies no loss of generality. The condition that the antecedent is not countervalid in $\mathcal{M}$ may be expressed by the condition that, for each element φ of Γ, $\updownarrow\varphi$ is not countervalid in $\mathcal{M}$.

6.4 *p*–Entailment and Logical Consequence

The link between our semantics and probabilistic semantics is easy to establish by means of the Adams' notion of *yielding* that is defined in terms of the Adams' notions explained in the precedent section. It may be easily adapted to our semantics and relativized to a model $\mathcal{M}$. Let $\Gamma \cup \{\varphi\}$ be a finite set of sentences of the form $(\psi \mid \chi)$ where χ and ψ are in nqc form and χ is not countervalid in $\mathcal{M}$. Γ *yields* φ in $\mathcal{M}$ iff (i) every truth-assignment $t_{\alpha}^{\mathcal{M}}$ that confirms Γ verifies φ, and (ii) every truth-assignment $t_{\alpha}^{\mathcal{M}}$ that falsifies no element of Γ does not falsify φ. Adams has proved that Γ *p*-entails φ iff either Γ is *p*-inconsistent, or there is a subset Γ' of Γ that yields φ. Adapting his proof to the preceding model-relative notions is straightforward.

Theorem 6.3

Let $\mathcal{M}$ be a model. Let $\Gamma \cup \{\varphi\}$ be a finite set of sentences syntactically simple sentences of the form $(\psi \mid \chi)$, where χ and ψ are in nqc form and φ is not countervalid in $\mathcal{M}$. We prove first (a) that if $\Gamma \models_{\mathcal{M}} \varphi$, then either Γ is unsatisfiable in $\mathcal{M}$ or some subset Γ' of Γ yields φ in $t_{\alpha}^{\mathcal{M}}$ and (b) that if Γ is unsatisfiable in $t_{\alpha}^{\mathcal{M}}$ or some subset Γ' of Γ yields φ in $t_{\alpha}^{\mathcal{M}}$, then $\Gamma \models_{\mathcal{M}} \varphi$.

Proof

(A) Suppose that $\Gamma \models_{\mathcal{M}} \varphi$. If $\models_{\mathcal{M}} \varphi$ or Γ is unsatisfiable, then trivially Γ yields φ, so that the theorem immediately is proved in this case. If $\Gamma = \emptyset$ and not $\models_{\mathcal{M}} \varphi$ then not $\Gamma \models_{\mathcal{M}} \varphi$. Suppose that $\Gamma \neq \emptyset$ so that there is a non-empty subset Γ' of Γ such that for every world α in $\mathcal{M}$ it holds: (i) for every $\psi \in \Gamma'$ it holds that not $\overline{\models}^{\mathcal{M}}_{\alpha}$ ψ then not $\overline{\models}^{\mathcal{M}}_{\alpha}$ φ, (ii) if for every $\psi \in \Gamma'$ it holds that not $\overline{\models}^{\mathcal{M}}_{\alpha}$ ψ and for at least one element ψ of Γ' if it holds that $\models^{\mathcal{M}}_{\alpha} \psi$ then it holds that $\models^{\mathcal{M}}_{\alpha} \varphi$. Consider the case (i). If for every $\psi \in \Gamma'$ it holds that not $\overline{\models}^{\mathcal{M}}_{\alpha}$ ψ then ψ is not falsified at α. And if it holds that not $\overline{\models}^{\mathcal{M}}_{\alpha}$ φ, then φ is not falsified at α. Consider case (ii). Suppose that for every $\psi \in \Gamma'$ it holds that not $\overline{\models}^{\mathcal{M}}_{\alpha}$ ψ and for at least one element ψ of Γ' it holds that $\models^{\mathcal{M}}_{\alpha} \psi$. Since $\models^{\mathcal{M}}_{\alpha} \varphi$, φ is either verified at α or ψ is a near-tautology in $\mathcal{M}$. If φ is a near-tautology in $\mathcal{M}$, then it is not falsified at every world, and there is a world at which it is verified. In this case $\emptyset$ yields φ and the condition is satisfied.

(B) Suppose that either Γ is unsatisfiable or some subset of Γ yields φ in $\mathcal{M}$. If Γ is unsatisfiable then $\Gamma \models_{\mathcal{M}} \varphi$ by definition. If $\Gamma = \emptyset$, then φ is such that no truth-assignment falsifies it and at least one verifies it. In this case φ is valid in $\mathcal{M}$, so that by definition 2.15 φ is an $\mathcal{M}$-consequence of Γ. Otherwise, let Γ' be a subset of Γ yielding φ. By hypothesis, every truth-assignment that falsifies no element of Γ does not falsify φ. Since every element which is not false at some world α in $\mathcal{M}$ is not falsified by the truth-assignment $t^{\mathcal{M}}_{\alpha}$, it follows that for every $\psi \in \Gamma'$ it holds that not $\overline{\models}^{\mathcal{M}}_{\alpha}$ ψ then it holds that not $\overline{\models}^{\mathcal{M}}_{\alpha}$ φ. Suppose now that, for every world α in $\mathcal{M}$, every truth-assignment $t^{\mathcal{M}}_{\alpha}$ confirming Γ' verifies φ. For every world α at which no elements of Γ' is falsified by the truth-assignment $t^{\mathcal{M}}_{\alpha}$ but some element ψ of Γ' is verified by $t^{\mathcal{M}}_{\alpha}$, it holds that $\models^{\mathcal{M}}_{\alpha} \psi$. Moreover, by hypothesis, also φ is verified by $t^{\mathcal{M}}_{\alpha}$. In this case it holds that $\models^{\mathcal{M}}_{\alpha} \varphi$. Suppose there is a world β at which it holds that $\models^{\mathcal{M}}_{\beta} \psi$ while every other element χ of Γ' is such that it holds that not $\overline{\models}^{\mathcal{M}}_{\beta}$ χ and suppose that ψ is not verified by the truth-assignment $t^{\mathcal{M}}_{\beta}$. In this case ψ is $\mathcal{M}$-valid. Let Δ be the subset of Γ' of those sentences of Γ' that are $\mathcal{M}$-valid. Consider the set $\Gamma'' = \Gamma' - \Delta$. The set of worlds at which the elements of Γ'' are true coincides with the set of worlds at which the elements of Γ'' are verified. Hence $\Gamma'' \models_{\mathcal{M}} \varphi$, and since $\Gamma'' \subseteq \Gamma$ it holds that $\Gamma \models_{\mathcal{M}} \varphi$.

q.e.d.

Theorem 6.3 does not consider the case in which every element of $\Gamma \cup \{\varphi\}$ is singular. However, in such a case, according to *our* definitions it holds that $\Gamma \models_{\mathcal{M}} \varphi$ iff Γ *p*-entails φ.

7. Assertion of Conditionals

'Assertion' denotes a pragmatic notion. Asserting a two-valued sentence φ (or rather, the proposition it expresses) is *categorically* asserting that φ is true. By contrast, asserting simple indicative conditionals amounts to making *conditional assertions*, that is to assert the consequent with the proviso that the antecedent is true. The assertion of the consequent is conditional in the sense that it expresses a conditional commitment. In uttering a (simple) conditional sentence, the utterer anticipates in advance that should the antecedent turn out to be false the assertion will be *eo ipso* cancelled. To capture the semantical aspects of conditional assertion, we have to move from the classical logic of sentences to the logic of tri-events. The idea that the proper assertive speech act relative to conditionals is a conditional assertion is not new. It was indeed implicit in Ramsey's and de Finetti's thought in the thirties—Ramsey (1928) 1990: 155 n.; de Finetti (1934) 2006: 103. It was half-heartedly advanced by Quine (1966: 12) and advocated by a few other authors after that—among them von Wright (1957), Belnap (1970), Mackie (1973), Edgington (2007: 176-80), De Rose-Grandy (1999).

The problem arises about the assertion of compounds of conditionals. If the assertion of two simple conditionals consists of two separate conditional assertions, in what sense the conjunction or the disjunction of them is a single conditional assertion? Since several antecedents are involved while conditional assertion involves a single antecedent, it seems that the view according to which the assertion of a tri-event is a conditional assertion seems to be inapplicable. This difficulty is, however, easily solved by theorems 5.4 and 5.14. Since the truth conditions of every tri-event are the same as a simple tri-event (that one may effectively find) and asserting the former amounts to asserting the latter, we may assume that asserting a tri-event is, in a general way, a single conditional assertion.[13] We may remark that in natural language, compounds of conditionals have little use. Admittedly, our theory, like other theories that allow compounds of conditional without restraint, extend the available syntactical combinations of natural language conditionals. This fact is typical of logical theories equipped with recursively defined connectives. Even the language of sentential logic extends the natural use of connectives borrowing from algebra the use of parentheses. Before Leibniz and Boole, logicians did not consider complex sentences built by Boolean connectives, and surely they are foreign to natural language. Nobody would understand the truth conditions of enough complex sentences of classical sentential logic without analysing them by the semantical rules. The situation with conditionals is parallel. If we adopt well-defined semantics, we must resort to the semantical rules to understand the truth conditions of enough complex sentences according to such semantics. Admittedly, this happens (in contrast with classical sentential logic) with much less complex compounds of conditional sentences. In any case, it is a matter of degree, not a fundamental difference.

Since there are several alternative semantics, the problem arises to select, among these alternatives, the one better suited in those few cases in which com-

[13] Here the problem of logical omniscience arises. However, this problem hits the application of every logical theory to knowledge and belief. The present account is no exception.

pound conditionals have a definite meaning in natural language. We will endeavour in this last direction in the next section. In the light of the above considerations, we assume, in the following discussion, that tri-events are simple conditionals with two-valued antecedents and consequents. To simplify further the picture, we also assume that no modal symbol occurs in them.

The idea of a truth-conditional *semantics* based on this view has received little acceptance. Lewis' Triviality Results and McGee's results have played a significant role in this attitude. Independently of these results, many philosophers find it difficult to accept that the pragmatic attitude of asserting a proposition with a caveat has a semantical counterpart in a conditional sentence. We reverse the view. Tri-events come first as logical objects *per se*, whose semantics is well defined in the framework of partial modal logic, and we maintain that the proper way to assert them is by conditional assertions.

However, there is also reluctance in accepting the idea of partial semantics concerning conditionals. Partial semantics, although prominent logicians like Kripke (1975) have used it in other contexts, looks like something "deviant", as it requires a departure from the prevailing view that sentences, including conditionals, are either true or false and *tertium non datur*. Another difficulty is that our approach seems to preclude a unified approach to conditionals, encompassing both indicative and counterfactual conditionals. Indeed, considering conditionals with false antecedents as being neither true nor false, seems *prima facie* to preclude a truth-conditional semantics for counterfactuals.

We insist on claiming that making a conditional assertion of a tri-event φ does not mean asserting that φ is true. One must not confuse the assertion of φ with the assertion of $\uparrow\varphi$, except when φ and $\uparrow\varphi$ are logically equivalent (that is when φ is a two-valued sentence). When one asserts φ is asserting that φ is true under the hypothesis that φ has a truth-value, which depends on the truth-value of a two-valued sentence. If that condition fails, nothing is asserted. This fact, pragmatically, implicates that the author of the assertion does not know the truth-value of the antecedent. In normal circumstances, the utterance of tri-event amounts to the utterance of an epistemic open conditional. The class of open conditionals largely overlaps the class of the so-called indicative conditionals (so that the non-explicit implicature is largely conventional). If, after the utterance of a conditional of this kind, the utterer becomes certain that the antecedent is true, she should be prepared to make a categorical assertion of both the consequent and the antecedent.

We may ask what is the proper speech act if the antecedent of an open conditional turns out to be false. There is no commitment to the truth of the consequent under the condition that the antecedent is false. So, suppose the antecedent turns out to be false. In that case, the *conditional commitment to the truth* of the consequent fails because it fails the condition under which that commitment would be at work according to the conditional assertion. Thus, the corresponding tri-event turns out to have no truth value.

8. Non-simple Conditionals

8.1 Negation

There is only one primitive negation (represented by the connective '$\neg$'). Denying a tri-event is denying its consequent while keeping the antecedent as an explicit supposition. As asserting that φ is a conditional assertion, denying that φ is a conditional denial. The conditional denial of a tri-event φ is just the conditional assertion of its negation. As we must not confuse the (categorical) assertion of $\uparrow\varphi$ with the conditional assertion of φ, we must not confuse the conditional denial of φ — that is the conditional assertion of $\neg\varphi$) — with the (categorical) assertion of $\uparrow\neg\varphi$, that is with the (categorical) assertion that φ is false. The latter logically entails the former without being equivalent to it. Nor we must confuse the act of conditionally denying that φ with categorically denying that φ is true, which is entailed by conditionally denying that φ but does not entail it. Of course, one may categorically assert that $\uparrow\varphi$ (that is, one may assert that φ is true) in response to the conditional assertion of $\neg\varphi$. This assertion entails the conditional denial of the conditional claim. Analogously, if one conditionally asserts that φ, the other party may categorically assert that $\uparrow\neg\varphi$ (that is that φ is false), which entails *a fortiori* conditionally denying that φ, that is conditionally asserting that $\neg\varphi$.

8.2 $\wedge$-Introduction Fails

The simultaneous assertion of two-valued sentences φ and ψ may be represented either by the set $\Gamma = \{\varphi, \psi\}$ or by the conjunction $(\varphi \wedge \psi)$. Indeed, Γ and $(\varphi \wedge \psi)$ share the same set of logical consequences. This result is a consequence of the following two deductive properties of sentential conjunction: $\wedge$-introduction and $\wedge$-elimination. According to $\wedge$-introduction $(\varphi \wedge \psi)$ is a logical consequence of Γ. According to $\wedge$-elimination both φ and ψ are logical consequences of $(\varphi\wedge\psi)$. Now, given two sentences whatsoever of our theory, $\wedge$-elimination holds in a general way. By contrast, if φ and ψ are not both two-valued, then $\wedge$-introduction fails.[14] This fact means that asserting (or believing) two tri-events simultaneously is not the same as asserting (or believing) their conjunction. Consider, for example, two tri-events of the form $(\psi \mid \varphi)$ and $(\psi \mid \neg\varphi)$. In this case the set $\Gamma = \{(\psi \mid \varphi), (\psi \mid \neg\varphi)\}$ may well be consistent or satisfiable, so that one may well assert both tri-events (in the conditional sense explained above). However, from Γ does not follows in general $\chi = (\psi \mid \varphi) \wedge (\psi \mid \neg\varphi)$. Indeed, if φ and ψ are logically independent two-valued sentences, χ is necessarily false.

There is no conjunction for which both the introduction rule and the elimination rule hold (see Adams 1998: 177, Schulz 2009). Intuitively, if φ and ψ are two-valued sentences, conditional asserting both that $(\psi \mid \varphi)$ and that $(\psi \mid \neg\varphi)$ would imply asserting that ψ (asserting that ψ conditionally to both φ and $\neg\varphi$ amounts to unconditionally asserting that ψ). Indeed, it holds that ψ is a logical consequence of $\{(\psi \mid \varphi), (\psi \mid \neg\varphi)\}$. So, in general, a set (rather than conjunction) represents the *deductive* content of two or more tri-events (meant as the set of their

[14] Alternative Sobociński conjunction and disjunction definable in three-valued logic are not suitable in our semantics since they do not satisfy associativity.

logical consequences). In applied logic, we may represent the simultaneous assertion of two or more conditionals by the set of those conditionals. Our semantics perfectly represents the set of the logical consequences of this speech act.

However, if one uses the conjunction instead, the intended meaning of the compound sentence should be determined by the adopted semantics. It is not correct to borrow it mechanically from two-valued sentence logic. If we look at natural language, there is little conjunction out of simultaneous assertion of conditionals. In the light of this consideration, we cannot use an iterative conjunction connective to represent that sentence whose assertion is equivalent to the assertion of both the conjuncts, simply because it does not exist. However, we may well obtain this use of the conjunction resorting to a set of conditional sentences.

On the other hand, the fact that our theory extends the natural semantics of conditionals, adding a "conjunction" of conditionals giving to it a meaning different from simultaneous assertion, is not, in our view, a good reason to reject the present theory as far as compounds of conditionals are concerned (let alone to conclude that compounds of conditionals lack truth conditions). After all, the new conjunction (even if foreign to ordinary language) may turn out to be useful in other contexts (especially in automated reasoning).

8.3 Sets as Disjunctions

In sentential logic, the relationship between disjunction and assertion is not analogous to conjunction and assertion. If φ and ψ express two-valued propositions, then asserting that either φ or ψ obtains does not entail either asserting φ or asserting ψ. One may assert $(\varphi \vee \psi)$ while asserting neither φ nor ψ. The same holds for tri-events. Is there a way to express that we are prepared to assert at least one of two or more tri-events? The answer is yes if the desired "disjunction" is expressed by a set of sentences, being the succedent of a sequent. So, in a sequent $\Gamma \Rightarrow \Delta$, Δ is a set of sentences such that at least one of them must be conditionally asserted to their respective antecedents if one asserts all the elements of Γ. The desired disjunction should be stronger than $\vee$, so that the sequent $(\varphi \vee \psi) \Rightarrow \{\varphi, \psi\}$ should not be valid.

8.4 Logical Consequence for Multiple Conclusions (Structural Sequents)

We have seen that our notion of logical consequence is coextensive with Adams *p*-entailment. Now, Adams developed also a sequent version of his probabilistic logic. Since sequents allow representing conjunction and disjunction in a manner that is different from conjunctions and disjunctions represented by connectives, the problem arises whether it is possible to provide a truth-conditional counterpart of his probabilistic definition coextensive with it. The answer is definite, as the following results will show.

Definition 8.1 (Valid Sequents)

The finite set Δ of sentences is an $\mathcal{M}$-consequence of the finite set of sentences Γ ($\Gamma \underset{\mathcal{M}}{\Rightarrow} \Delta$) iff either

(a) There is a non-empty subset Γ' of Γ such that at every world α in $\mathcal{M}$ there is an element φ of Γ' such that $\overline{\models}^{\mathcal{M}}_{\alpha}\ \varphi$
or

(b) there is a subset Γ' of Γ and a non-empty n-tuple $\Delta' = \langle \varphi_1, \ldots, \varphi_n \rangle$ of elements of Δ such that the following conditions are satisfied:

  (i) for every world α in $\mathcal{M}$ such that no element of Γ' is false at α and at least one element of Γ' is true at α it holds that at least one element of Δ' is true at α;

  (ii) for every world α in $\mathcal{M}$, such that no element of Γ' is false at α and for some i ($1 \leq i \leq n$) either $\models^{\mathcal{M}}_{\alpha}\ \varphi_i$ or $\overline{\models}^{\mathcal{M}}_{\alpha}\ \varphi_i$ while for all $j < i$ (if any) not $\models^{\mathcal{M}}_{\alpha}\ \varphi_j$ and not $\overline{\models}^{\mathcal{M}}_{\alpha}\ \varphi_j$, it holds that $\models^{\mathcal{M}}_{\alpha}\ \varphi_i$;

or

(c) $\Gamma \cup \Delta$ is singular in $\mathcal{M}$.

Definition 8.2 (Sequent *p*-entailment)

The set of sentences Γ p-entails the set of sentences Δ in the model $\mathcal{M}$ iff either

(a) for every $\epsilon > 0$ there is a $\delta > 0$ such that every probability function $\mathbf{P}^{\mathcal{M}}$ defined over $\mathcal{M}$ according to which every element $\psi \in \Gamma$ such that $\mathbf{P}^{\mathcal{M}}(\updownarrow \psi) > 0$ has a probability value $\geq \delta$ assigns a probability value $\geq \epsilon$ to at least one element of Δ
or

(b) $\Gamma \cup \Delta$ is singular in $\mathcal{M}$.

Conditions (c) in definition 8.1 and (b) in definition 8.2 were not present in Adams original definition. We have added them because, in our system, singular elements are present but have no probability value. At the same time, in Adams 1986 it is assumed that such conditionals have probability 1 (so that such conditions follow from the other conditions). In Adams 1975 they are not present in the language.

Now we are in the position of proving that our definition of sequent validity is coextensive to Adams' sequent p-entailment. To prove this, we resort, like in the case of single conclusion inferences to Adams 1986 and Bamber's work (1994). Then, adapting from Bamber, slightly correcting Adams' original yielding for p-entailment for sequents, we define it in the following way:

Definition 8.3 (Sequent Yielding)

Let Γ and Δ be finite sets of sentences of the form $(\psi \mid \chi)$ where χ and ψ are in nqc form and χ is not countervalid. Γ *yields* Δ in $\mathcal{M}$ iff either

(A) Δ is empty and for every element φ of Γ, φ is verified at no world in $\mathcal{M}$ by the truth-assignment $t^{\mathcal{M}}_{\alpha}$ (as explained above) and is falsified by $t^{\mathcal{M}}_{\alpha}$ at some world in $\mathcal{M}$
or

(B) Δ is not empty and the following conditions are satisfied:

(i) every truth assignment $t^{\mathcal{M}}_{\alpha}$ (if any) that does not falsifies any element of Γ and verifies at least one element of Γ verifies at least one element of Δ;

(ii) there is an order $\preccurlyeq$ on the set Δ such that if for every truth-assignment $t^{\mathcal{M}}_{\alpha}$ there is at least one element of Δ which $t^{\mathcal{M}}_{\alpha}$ either verifies or falsifies then $t^{\mathcal{M}}_{\alpha}$ verifies the first of such elements according to the order $\preccurlyeq$ (the preceding ones by $\preccurlyeq$ being neither verified nor falsified by $t^{\mathcal{M}}_{\alpha}$).

Bamber (1994) proves the following result (which we adapt here to our model-theoretic approach):

Theorem 8.1

Let $\mathcal{M}$ be a model. Γ p-entails Δ in $\mathcal{M}$ iff there is a subset Γ' of Γ and a subset Δ' of Δ such that Γ' yields Δ'.

Theorem 8.1 allows proving in our truth-conditional semantics the coextension of p-entailment and validity for sequents.

Theorem 8.2

Let $\mathcal{M}$ be a model. Let $\Gamma \cup \Delta$ be a finite set of sentences syntactically simple sentences of the form $(\psi \mid \chi)$, where χ and ψ are in nqc form and φ is not countervalid in $\mathcal{M}$. Then Γ p-entails Δ in $\mathcal{M}$ iff it holds that $\Gamma \underset{\mathcal{M}}{\Rightarrow} \Delta$.

Proof

Suppose that Γ p-entails Δ in $\mathcal{M}$. By thorem 8.1 there are two sets $\Gamma' \subseteq \Gamma$ and $\Delta' \subseteq \Delta$ such that Γ' yields Δ'. In the case (a) of definition 8.3, at every world α in $\mathcal{M}$ there is an element φ of Γ' such that $\overline{\models}^{\mathcal{M}}_{\alpha}$ φ. Now either φ is falsified by the truth-assignment $t^{\mathcal{M}}_{\alpha}$ or φ is neither falsified nor verified by $t^{\mathcal{M}}_{\alpha}$ but is falsified at some world β in $\mathcal{M}$. Now, in this case Γ' yields $\emptyset$, so that Γ p-entails Δ in the model $\mathcal{M}$. In the case (b) of definition 8.3, suppose that Γ' and Δ' be n-tuples both satisfying conditions (i)-(iii) of definition 8.3. Suppose that for a certain world α in $\mathcal{M}$ for no element φ of Γ' it holds that $\overline{\models}^{\mathcal{M}}_{\alpha}$ φ and for at least one element ψ of Γ' it holds that $\models^{\mathcal{M}}_{\alpha}$ ψ. Every sentence that is not true at α is not verified (and not falsified) by $t^{\mathcal{M}}_{\alpha}$. Let φ be an element of Γ' such that $\models^{\mathcal{M}}_{\alpha} \varphi$. Since, by hypothesis, for every world γ in $\mathcal{M}$ it holds that not $\overline{\models}^{\mathcal{M}}_{\gamma} \varphi$, φ is a near-tautology. Suppose that φ is not verified by $t^{\mathcal{M}}_{\alpha}$. Then there is a world β at which no element of Γ' is falsified and at which φ is verified. Since φ is true at β, by hypothesis there is a subset Δ'' of Δ' such that at least one element ψ of Δ'' is true at β. Again either ψ is verified at β, which would prove that every truth assignment $t^{\mathcal{M}}_{\alpha}$ that does not falsify any element of Γ' and verifies at least one element of Γ' verifies at least one element of Δ'' or ψ is a near-tautology. But in this last case, there is a world γ in $\mathcal{M}$ at which ψ is verified, and no world in $\mathcal{M}$ at which ψ is falsified. In this case $\{\psi\}$ is yielded by $\emptyset$, which is contained in Δ' so that Γ' yields Δ'. So, if condition (i) is satisfied then either every truth assignment $t^{\mathcal{M}}_{\alpha}$ that does not falsify any element of Γ' and verifies at least one element of Γ' verifies at least one element of Δ' or, in any case, Γ *p*-entails Δ. Assume now that condition (ii) of definition 8.3 is satisfied. Suppose that for every world α in $\mathcal{M}$, no element of Γ' is false at α and for some $i(1 \leq i \leq n)$ either $\models^{\mathcal{M}}_{\alpha} \varphi_i$ or $\overline{\models}^{\mathcal{M}}_{\alpha} \varphi_i$ while for all $j < i$(if any) not $\models^{\mathcal{M}}_{\alpha} \varphi_j$ and not $\overline{\models}^{\mathcal{M}}_{\alpha} \varphi_j$. In the case where for no element φ of Γ', it holds that $\overline{\models}^{\mathcal{M}}_{\alpha} \varphi$, if it holds that not $\models^{\mathcal{M}}_{\alpha} \varphi$ and not $\overline{\models}^{\mathcal{M}}_{\alpha} \varphi$, φ is neither verified nor falsified by $t^{\mathcal{M}}_{\alpha}$. If for some element φ of Γ' it holds that $\models^{\mathcal{M}}_{\alpha} \varphi$, φ is either verified at α or it is a near-tautology in $\mathcal{M}$. in the first case let us consider the first element ψ of Δ' such that it holds either that $\models^{\mathcal{M}}_{\alpha} \psi$ or $\overline{\models}^{\mathcal{M}}_{\alpha} \psi$ while for each preceding sentence χ (if any), according to $\preccurlyeq$ it holds that are neither $\models^{\mathcal{M}}_{\alpha} \chi$ nor $\overline{\models}^{\mathcal{M}}_{\alpha} \chi$. By hypothesis ψ is true at α. Now, in this case ψ is either verified or it is a near-tautology in $\mathcal{M}$. In the first case it holds that there is an order $\preccurlyeq$ on the set $\{\varphi_1, \ldots, \varphi_n\}$ such that if for every truth-assignment $t^{\mathcal{M}}_{\alpha}$ there is at least one element of Δ such that $t^{\mathcal{M}}_{\alpha}$ either verifies or falsifies then $t^{\mathcal{M}}_{\alpha}$ verifies the first of such elements according to the order $\preccurlyeq$ (the preceding ones by $\preccurlyeq$ being neither verified nor falsified by $t^{\mathcal{M}}_{\alpha}$). If ψ is a near-tautology then $\emptyset$ yields $\{\psi\}$, so that Γ *p*-entails Δ. Let Γ'' be the set of the elements of Γ which are not near-tautologies. Every element φ of Γ'' (which may be empty) such that it holds that $\models^{\mathcal{M}}_{\alpha} \varphi$ is verified at α by $t^{\mathcal{M}}_{\alpha}$. But in this case, by what has been said, Γ'' yields Δ, so that Γ *p*-entails Δ. The case (b) in definition 8.3 is impossible by the conditions of the theorem.

q.e.d.

8.5 Switches Paradox

The switches paradox is due to Adams that formulates it in the following way:

> If switches *A* and *B* are thrown the motor will start. Therefore, either if switch *A* is thrown the motor will start or if switch *B* is thrown the motor will start (Adams 1975: 32).

Nobody would consider this argument compelling. However, if the material conditional is used to formalise it, it turns out to be valid. So, material conditional appears to be unable to deal appropriately in this case. The paradox evaporates if modal theories (strict implication theory or Stalnaker-Lewis theories) are used instead of the material conditional. The problem arises if our theory can solve the paradox.

P. Milne (1997: 224) writes that "with the de Finetti-Goodman-Nguyen conjunction and disjunction conditional assertions/events provide no escape from the switches paradox". That result is true for the original de Finetti's theory equipped with the natural consequence relation based on the lattice order. It would be true also for our theory if the conclusion in disjunctive form were expressed using $\vee$ (rather than a set in the succedent of a sequent.) However, with this last move, the switches paradox is easily solved. In fact, the sequent $\{(\chi \mid (\varphi \wedge \psi))\}, \Gamma \underset{\mathcal{M}}{\Rightarrow} \Delta, \{\chi \mid \varphi, \chi \mid \psi\}$ is not valid. Notice that this sequent is also not *p*-valid so that Adams theory offers a solution to the switches paradox as well. Our solution is not in contrast with Adams' solution since our notion of logical consequence is coextensive with *p*-entailment. Rather, it provides truth-conditional foundations of Adams logic. Indeed, in our theory, the notion of logical consequence is defined throughout in truth-conditional terms.

8.6 Sequent Antecedents and Succedents concerning Connectives

For representing conjunctions and disjunctions, sets are different from connectives. They cannot be combined freely with the connectives, and recursive formation rules do not govern them. The use of sets makes sense only in an argument where the set of those premises represents a conjunctive premise, and the set of those conclusions represents a disjunctive conclusion. In sharp contrast with standard sequent theory, sequent conjunction and disjunction do not coincide with the respective connectives. For example, concerning $\wedge$ and $\vee$ connectives, the sequent conjunction is weaker than $\wedge$, and the sequent disjunction is stronger than $\vee$. Indeed, the following sequent schemas are valid:

$$\frac{(\phi \wedge \psi) \Rightarrow \chi}{\{\phi, \psi\} \Rightarrow \chi} \qquad \frac{\chi \Rightarrow \{\phi, \psi\}}{\chi \Rightarrow (\phi \vee \psi)}$$

while the inverse schemas are not

$$\frac{\{\phi, \psi\} \Rightarrow \chi}{(\phi \wedge \psi) \Rightarrow \chi} \qquad \frac{\chi \Rightarrow (\phi \vee \psi)}{\chi \Rightarrow \{\phi, \psi\}}.$$

8.7 Modus Ponens Fails

Modus ponens schema is not valid in our theory. This invalidity is not a defect of the present theory. There is an important result by McGee (1985) about *modus ponens* which shows that *modus ponens* may not preserve truth concerning iterated conditionals. In light of this result, it is surely not a defect of our theory if *modus ponens* is not valid in it except for simple conditionals. Since we know that every tri-event is logically equivalent to a simple tri-event, translating the sentences in an instance of the *modus ponens*, the schema does not yield necessarily another instance of the *modus ponens* schema. Theorems 5.5 and 5.14 show that the simple counterpart of a tri-event has, in general, a S5 antecedent and a S5 consequent.

However, as McGee's example shows, our intuitions about non-simple conditionals cannot be mechanically borrowed by sentence logic. *Modus ponens* is very important in sentence logic concerning material conditional. However, it would be a fallacy to employ it in inferences involving non-simple conditionals without further constraints that render *modus ponens* a valid inference. It also should be stressed that since valid sentences are essentially two-valued in our theory, one may safely adopt *modus ponens* as a proof-theoretic rule in a possible axiomatic system of tri-events (whose development is part of a future agenda).

8.8 Conditional Excluded Middle

Excluded middle cannot hold in our theory because tri-events are not two-valued sentences, so that $(\varphi \vee \neg\varphi)$ is not a valid schema. From this it follows also that *conditional* excluded middle is not valid. For, if $\varphi = (\chi \mid \psi)$ then $\neg\varphi = (\neg\chi \mid \psi)$, so that also $(\chi \mid \psi) \vee (\neg\chi \mid \psi)$ is not valid. However, in a general way, if φ is not singular (so that it may have a truth-value) $(\varphi \vee \neg\varphi)$ is true at every world. That is, excluded middle (including conditional excluded middle) is a near-tautology. Indeed, every classic sentential tautology is a near-tautology according to our semantics.

When the conclusion of an argument has many conclusions (that is, in applied logic, when the conclusion asserts at least one disjunct in the presence of the premises), Adams proved an interesting result. According to it, sequents, once translated in the language of Lewis' modal theory of conditionals, where a sequent of simple conditionals $\{\varphi_1, \ldots, \varphi_n\} \Rightarrow \{\psi_1, \ldots, \psi_m\}$ is translated as $(\varphi_1 \wedge \cdots \wedge \varphi_n) \rightarrow (\psi_1 \vee \ldots \vee \psi_m)$ are *p*-valid iff their translation is valid in the Lewis' VW system (Adams 1986: 269). Since *p*-entailment is coextensive to our logical consequence or sequent validity, this result also holds in our semantics.

8.9 Import-Export

It is well known that the Stalnaker-Thomason's and Lewis' semantics do not satisfy the import-export law for conditionals. Lycan (2001: 82), commenting on Gibbard 1981 notices that the import-export principle entails the logical equivalence between $(A > (B > A)$ and $((A\&B) > A)$ (where '&' stands for 'and' and '>' stands for the conditional). Moreover, since $((A\&B) > A)$ is considered as valid, $(A > (B > A)$ would be valid as well. However, the following example shows that $(A > (B > A))$ is not always true:

> If Harry runs fifteen miles this afternoon, then if he is killed in a swimming accident this morning, he will run fifteen miles this afternoon.

In this case, A ("Harry runs fifteen miles this afternoon") and B ("he is killed in a swimming accident this morning") are such that their conjunction is impossible, and this logical situation generates the paradox. Now, let us examine this example in the light of our modal theory of tri-events. We discover that the schema $((\psi \mid \varphi) \mid \psi)$, where φ and ψ are two-valued sentences, is *not* always valid, but that the only exception is just when $(\varphi \wedge \psi)$ is impossible. The following schema, if φ and ψ are ordinary sentences, is valid:

$\Diamond(\varphi \wedge \psi) \rightarrow ((\psi \mid \varphi) \mid \psi)$

More generally, the following theorem holds:

Theorem 8.3

Let $\hat{\varphi}$, $\hat{\psi}$, and $\hat{\chi}$ ordinary sentences. Then the following schema is O-valid:
(RLIE) $\Diamond(\hat{\varphi} \wedge \hat{\psi}) \rightarrow (((\hat{\chi} \mid \hat{\psi}) \mid \hat{\varphi}) \leftrightarrow (\hat{\chi} \mid (\hat{\varphi} \wedge \hat{\psi})))$

Proof

Suppose that $\models \Diamond(\hat{\varphi} \wedge \hat{\psi})$. Let $\mathcal{M}$ be any model. Since $\hat{\varphi}$ and $\hat{\psi}$ are, by theorem 4.5 essentially two-valued sentences, this means that there is world α in $\mathcal{M}$ it holds that $\models^{\mathcal{M}}_{\alpha} (\hat{\varphi} \wedge \hat{\psi})$, and therefore that $\models^{\mathcal{M}}_{\alpha} \hat{\varphi}$ and $\models^{\mathcal{M}}_{\alpha} \hat{\psi}$. Under this assumption and under theorem 4.3, we have to prove:

(a) if it holds that $\models^{\mathcal{M}}_{\alpha} ((\hat{\chi} \mid \hat{\psi}) \mid \hat{\varphi})$ then $\models^{\mathcal{M}}_{\alpha} (\hat{\chi} \mid (\hat{\varphi} \wedge \hat{\psi}))$;

(b) If it holds that $\models^{\mathcal{M}}_{\alpha} (\hat{\chi} \mid (\hat{\varphi} \wedge \hat{\psi}))$ then $\models^{\mathcal{M}}_{\alpha} ((\hat{\chi} \mid \hat{\psi}) \mid \hat{\varphi})$;

(c) If it holds that $\overline{\models}^{\mathcal{M}}_{\alpha} ((\hat{\chi} \mid \hat{\psi}) \mid \hat{\varphi})$ then $\overline{\models}^{\mathcal{M}}_{\alpha} (\hat{\chi} \mid (\hat{\varphi} \wedge \hat{\psi}))$;

(d) If it holds that $\overline{\models}^{\mathcal{M}}_{\alpha} (\hat{\chi} \mid (\hat{\varphi} \wedge \hat{\psi}))$ then $\overline{\models}^{\mathcal{M}}_{\alpha} ((\hat{\chi} \mid \hat{\psi}) \mid \hat{\varphi})$;

(a) Let $\mathcal{M}$ be any model and let α any world in $\mathcal{M}$. Suppose that $\models^{\mathcal{M}}_{\alpha} ((\hat{\chi} \mid \hat{\psi}) \mid \hat{\varphi})$. By definition 2.4, either (j) $\models^{\mathcal{M}}_{\alpha} (\hat{\chi} \mid \hat{\psi})$ and $\models^{\mathcal{M}}_{\alpha} \hat{\varphi}$ or (jj) there is a world β in $\mathcal{M}$ at which it holds that $\models^{\mathcal{M}}_{\beta} (\hat{\chi} \mid \hat{\psi})$ and $\models^{\mathcal{M}}_{\beta} \hat{\varphi}$ and at no world γ in $\mathcal{M}$ it holds that $\models^{\mathcal{M}}_{\gamma} \hat{\varphi}$ and $\overline{\models}^{\mathcal{M}}_{\gamma} (\hat{\chi} \mid \hat{\psi})$. In the case (j), by definition 2.4, there are two sub-cases: (j1) $\models^{\mathcal{M}}_{\alpha} \hat{\chi}$ and $\models^{\mathcal{M}}_{\alpha} \hat{\psi}$, and (j2) there is a world β in $\mathcal{M}$ at which it holds that $\models^{\mathcal{M}}_{\beta} \hat{\chi}$ and $\models^{\mathcal{M}}_{\beta} \hat{\psi}$ and at no world γ in $\mathcal{M}$ it holds that $\models^{\mathcal{M}}_{\gamma} \hat{\psi}$ and $\overline{\models}^{\mathcal{M}}_{\gamma} \hat{\chi}$. In the case (j1), by definition 2.4 it holds that $\models^{\mathcal{M}}_{\alpha} (\hat{\chi} \mid (\hat{\varphi} \wedge \hat{\psi}))$, since it holds that $\models^{\mathcal{M}}_{\alpha} \hat{\varphi}$, $\models^{\mathcal{M}}_{\alpha} \hat{\psi}$, and $\models^{\mathcal{M}}_{\alpha} \hat{\chi}$. In the case (j2) since, by hypothesis, there is a world β in $\mathcal{M}$ at which $\models^{\mathcal{M}}_{\beta} (\hat{\varphi} \wedge \hat{\psi})$ and at no world γ in $\mathcal{M}$ it holds that $\models^{\mathcal{M}}_{\gamma} \hat{\psi}$ and $\overline{\models}^{\mathcal{M}}_{\gamma} \hat{\chi}$. *A fortiori* at no world γ in $\mathcal{M}$ it holds that $\models^{\mathcal{M}}_{\gamma} (\hat{\varphi} \wedge \hat{\psi})$ and $\overline{\models}^{\mathcal{M}}_{\gamma} \hat{\chi}$. It follows that $\models^{\mathcal{M}}_{\alpha} (\hat{\chi} \mid (\hat{\varphi} \wedge \hat{\psi}))$. In the case (jj), there is no world γ in $\mathcal{M}$ at which $\models^{\mathcal{M}}_{\gamma} \hat{\varphi}$ and $\overline{\models}^{\mathcal{M}}_{\gamma} (\hat{\chi} \mid \hat{\psi})$. In this case, there is no world γ at which $\models^{\mathcal{M}}_{\gamma} \hat{\varphi} \wedge \hat{\psi}$ and $\overline{\models}^{\mathcal{M}}_{\gamma} \hat{\chi}$, while there is a world β in $\mathcal{M}$ at which $\models^{\mathcal{M}}_{\beta} \hat{\varphi} \wedge \hat{\psi}$ and $\models^{\mathcal{M}}_{\beta} \hat{\chi}$. By definition 2.4 it holds that $\models^{\mathcal{M}}_{\alpha} (\hat{\chi} \mid (\hat{\varphi} \wedge \hat{\psi}))$.

(b) Let $\mathcal{M}$ be any model and let α any world in $\mathcal{M}$. Suppose that $\models^{\mathcal{M}}_{\alpha} (\hat{\chi} \mid (\hat{\varphi} \wedge \hat{\psi}))$. By definition 2.4, either (j) $\models^{\mathcal{M}}_{\alpha} \hat{\chi}$ and $\models^{\mathcal{M}}_{\alpha} (\hat{\varphi} \wedge \hat{\psi})$ or (jj) there is a world β in $\mathcal{M}$ at which it holds that $\models^{\mathcal{M}}_{\beta} \hat{\chi}$ and $\models^{\mathcal{M}}_{\beta} (\hat{\varphi} \wedge \hat{\psi})$ and at no world γ it holds that $\models^{\mathcal{M}}_{\gamma} (\hat{\varphi} \wedge \hat{\psi})$ and $\overline{\models}^{\mathcal{M}}_{\gamma} \hat{\chi}$. In the case (j), it holds that $\models^{\mathcal{M}}_{\alpha} \hat{\chi}$, $\models^{\mathcal{M}}_{\alpha} \hat{\varphi}$, and $\models^{\mathcal{M}}_{\alpha} \hat{\psi}$, so that, by definition 2.4 it holds that $\models^{\mathcal{M}}_{\alpha} ((\hat{\chi} \mid \hat{\psi}) \mid \hat{\varphi})$. In the case (jj), there is a world β in $\mathcal{M}$ at which $\models^{\mathcal{M}}_{\beta} \hat{\chi}$,

$\models^{\mathcal{M}}_{\beta} \hat{\psi}$, and $\models^{\mathcal{M}}_{\beta} \hat{\varphi}$ and at no world γ in $\mathcal{M}$ it holds that $\models^{\mathcal{M}}_{\gamma} \hat{\psi}$, $\models^{\mathcal{M}}_{\gamma} \hat{\varphi}$, and $\overline{\models}^{\mathcal{M}}_{\gamma} \hat{\chi}$. By definition 2.4 it holds that $\models^{\mathcal{M}}_{\alpha} ((\hat{\chi} \mid \hat{\psi}) \mid \hat{\varphi})$.

(c) Let $\mathcal{M}$ be any model and let α any world in $\mathcal{M}$. Suppose that $\overline{\models}^{\mathcal{M}}_{\alpha} ((\hat{\chi} \mid \hat{\psi}) \mid \hat{\varphi})$. By definition 2.4, either (j) $\overline{\models}^{\mathcal{M}}_{\alpha} (\hat{\chi} \mid \hat{\psi})$ and $\models^{\mathcal{M}}_{\alpha} \hat{\varphi}$ or (jj) there is a world β in $\mathcal{M}$ at which it holds that $\overline{\models}^{\mathcal{M}}_{\beta} (\hat{\chi} \mid \hat{\psi})$ and $\models^{\mathcal{M}}_{\beta} \hat{\varphi}$ and at no world γ in $\mathcal{M}$ it holds that $\models^{\mathcal{M}}_{\gamma} \hat{\varphi}$ and $\models^{\mathcal{M}}_{\gamma} (\hat{\chi} \mid \hat{\psi})$. In the case (j), by definition 2.4, there are two sub-cases: (j1) $\overline{\models}^{\mathcal{M}}_{\alpha} \hat{\chi}$ and $\models^{\mathcal{M}}_{\alpha} \hat{\psi}$, and (j2) there is a world β in $\mathcal{M}$ at which it holds that $\overline{\models}^{\mathcal{M}}_{\beta} \hat{\chi}$ and $\models^{\mathcal{M}}_{\beta} \hat{\psi}$ and at no world γ in $\mathcal{M}$ it holds that $\models^{\mathcal{M}}_{\gamma} \hat{\psi}$ and $\models^{\mathcal{M}}_{\gamma} \hat{\chi}$. In the case (j1), by definition 2.4 it holds that $\overline{\models}^{\mathcal{M}}_{\alpha} (\hat{\chi} \mid (\hat{\varphi} \wedge \hat{\psi}))$, since it holds that $\models^{\mathcal{M}}_{\alpha} \hat{\varphi}$, $\models^{\mathcal{M}}_{\alpha} \hat{\psi}$, and $\overline{\models}^{\mathcal{M}}_{\alpha} \hat{\chi}$. In the case (j2) since, by hypothesis, there is a world β in $\mathcal{M}$ at which $\models^{\mathcal{M}}_{\beta} (\hat{\varphi} \wedge \hat{\psi})$ and at no world γ in $\mathcal{M}$ it holds that $\models^{\mathcal{M}}_{\gamma} \hat{\psi}$ and $\models^{\mathcal{M}}_{\gamma} \hat{\chi}$. *A fortiori* at no world γ in $\mathcal{M}$ it holds that $\models^{\mathcal{M}}_{\gamma} (\hat{\varphi} \wedge \hat{\psi})$ and $\models^{\mathcal{M}}_{\gamma} \hat{\chi}$. It follows that $\models^{\mathcal{M}}_{\alpha} (\hat{\chi} \mid (\hat{\varphi} \wedge \hat{\psi}))$. In the case (jj) there is no world γ in $\mathcal{M}$ at which $\models^{\mathcal{M}}_{\gamma} \hat{\varphi}$ and $\models^{\mathcal{M}}_{\gamma} (\hat{\chi} \mid \hat{\psi})$. In this case, there is no world γ at which $\models^{\mathcal{M}}_{\gamma} \hat{\varphi}$, $\models^{\mathcal{M}}_{\gamma} \hat{\psi}$ and $\models^{\mathcal{M}}_{\gamma} \hat{\chi}$, while there is a world β in $\mathcal{M}$ at which $\models^{\mathcal{M}}_{\beta} \hat{\varphi}$, $\models^{\mathcal{M}}_{\beta} \hat{\psi}$ and $\overline{\models}^{\mathcal{M}}_{\beta} \hat{\chi}$. By definition 2.4 it holds that $\overline{\models}^{\mathcal{M}}_{\alpha} (\hat{\chi} \mid (\hat{\varphi} \wedge \hat{\psi}))$.

(d) Let $\mathcal{M}$ be any model and let α any world in $\mathcal{M}$. Suppose that $\overline{\models}^{\mathcal{M}}_{\alpha} (\hat{\chi} \mid (\hat{\varphi} \wedge \hat{\psi}))$. By definition 2.4, either (j) $\overline{\models}^{\mathcal{M}}_{\alpha} \hat{\chi}$ and $\models^{\mathcal{M}}_{\alpha} (\hat{\varphi} \wedge \hat{\psi})$ or (jj) there is a world β in $\mathcal{M}$ at which it holds that $\overline{\models}^{\mathcal{M}}_{\beta} \hat{\chi}$ and $\models^{\mathcal{M}}_{\beta} (\hat{\varphi} \wedge \hat{\psi})$ and at no world γ it holds that $\models^{\mathcal{M}}_{\gamma} (\hat{\varphi} \wedge \hat{\psi})$ and $\models^{\mathcal{M}}_{\gamma} \hat{\chi}$. In the case (j), it holds that $\overline{\models}^{\mathcal{M}}_{\alpha} \hat{\chi}$, $\models^{\mathcal{M}}_{\alpha} \hat{\varphi}$, and $\models^{\mathcal{M}}_{\alpha} \hat{\psi}$, so that, by definition 2.4 it holds that $\overline{\models}^{\mathcal{M}}_{\alpha} ((\hat{\chi} \mid \hat{\psi}) \mid \hat{\varphi})$. In the case (jj), there is a world β in $\mathcal{M}$ at which $\overline{\models}^{\mathcal{M}}_{\beta} \hat{\chi}$, $\models^{\mathcal{M}}_{\beta} \hat{\psi}$, and $\models^{\mathcal{M}}_{\beta} \hat{\varphi}$ and at no world γ in $\mathcal{M}$ it holds that $\models^{\mathcal{M}}_{\gamma} \hat{\psi}$, $\models^{\mathcal{M}}_{\gamma} \hat{\varphi}$, and $\models^{\mathcal{M}}_{\gamma} \hat{\chi}$. By definition 2.4 it holds that $\overline{\models}^{\mathcal{M}}_{\alpha} ((\hat{\chi} \mid \hat{\psi}) \mid \hat{\varphi})$.

q.e.d.

So, as far as Import-Export Law is concerned, our theory appears to be better suited than Stalnaker-Thomason's and Lewis' semantics. It avoids the difficulty raised by Lycan while keeping the law, albeit in a properly restricted form.

9. Conclusion

In this paper, I have presented, moving from de Finetti's idea about the so-called tri-events, a new modal semantics of conditionals. This semantics encompasses

Adams' probabilistic semantics and is coextensive to it as simple conditionals are concerned. Compounds of conditionals are logically equivalent to simple conditionals. We provide generalised probability axioms, which come down to the usual axioms for finite probability when only ordinary sentences are involved. The new theory bypasses Lewis' Triviality Results since the probability of a conditional is always the conditional probability of the consequent, given the antecedent. This result is possible because (a) non-simple conditionals obey the generalised axioms of probability, not to the standard axioms (the underlying algebraic structure being a lattice but not a Boolean algebra), and (b) we drop the central premise underlying the triviality results, according to which conditionals are two-valued sentences.

The Kripke-style modal semantics presented in this paper is a *partial semantics* so that there are distinct conditions for truth and falsehood. Moreover, in this theory, binary connectives, except material implication, which returns a two-valued sentence, have a *modal import*: their semantics depends on the totality of worlds. However, no modal import is present when mutually independent sentences are involved.

Adams' *p*-entailment may be applied unmodified to probability functions defined over the lattice of tri-events. Every sentence is logically equivalent to a simple conditional, where both antecedent and consequent are S5 sentences. We defined probability concerning a model $\mathcal{M}$. The probability of a tri-event is always equal to the ratio between the probability of two S5 sentences, provided the latter probability is greater than 0. There is no difficulty in dealing with the probability of S5 sentences since modal sentences and their negations (i.e. sentences of the form $\Diamond\varphi$, $\Box\varphi$, $\neg\Diamond\varphi$, and $\neg\Box\varphi$) in a single model behave like tautologies if true and like contradictions if false so that they have probability 1 or 0 at every world in a given model. Now, we have defined a truth-conditional consequence relation that is coextensive to *p*-entailment. This result challenges Adams tenet that conditionals have no truth conditions. Moreover, the probability of a tri-event is always the *conditional expectation* of the consequent given the antecedent. We also challenge Adams-like views that understand the probability of conditionals as the degree of assertability or acceptability without any connection with truth-values.

References

Adams, E.W. (1965), "On the Logic of Conditionals", *Inquiry*, 8, 166-97.

Adams, E.W. (1966), "Probability and the Logic of Conditionals", in Suppes, P. (ed.), *Aspects of Inductive Logic*, Amsterdam: North-Holland, 265-316.

Adams, E.W. (1975), *The Logic of Conditionals*, Dordrecht: Reidel.

Adams, E.W. (1977), "A Note on Comparing Probabilistic and Modal Logics of Conditionals", *Theoria*, 43, 186-94.

Adams, E.W. (1986), "On the Logic of High Probability", *Journal of Philosophical Logic*, 15, 255-79.

Adams, E.W. (1995), "Remarks on a Theorem of McGee", *Journal of Philosophical Logic*, 24, 343-48.

Adams, E.W. (1998), *A Primer of Probability Logic*, Stanford: CLSI Publications.

Bamber, D. (1994), "Probabilistic Entailment of Conditionals by Conditionals", *IEEE Transactions on Systems, Man, and Cybernetics*, 24, 1714-23.

Baratgin, J. (2021), "Discovering Early de Finetti's Writings on Trivalent Theory of Conditionals", *Argumenta*, this issue.

Belnap, N.D. (1970), "Conditional Assertion and Restricted Quantification", *Noûs*, 4, 1-12.

Bochvar, D. (1937) (1981), "On a Three-Valued Logical Calculus and Its Application to the Analysis of the Paradoxes of the Classical Extended Functional Calculus", trans. by Bergmann, M., *History and Philosophy of Logic*, 2, 87-112.

Chellas, B.F. (1980), *Modal Logic: An Introduction*, Cambridge: Cambridge University Press.

de Finetti, B. (1927-1935), *Logica plurivalente (1927-1935)*, de Finetti Papers, III. Research, 1927-2000 (Box 6, Folder 2), Pittsburgh: University of Pittsburgh (ULS Digital Collections), URL: https://digital.library.pitt.edu/islandora/object/pitt:31735033466552/viewer.

de Finetti, B. (1934) (2006), *L'invenzione della verità*, Milano: Raffaello Cortina.

de Finetti, B. (1935) (1995), "The Logic of Probability", *Philosophical Studies*, 77, 181-90.

de Finetti, B. (1936), "Les probabilités nulles", *Bulletin de Sciences Mathematiques*, 275-88.

de Finetti, B. (1979) (2008), "Prevision, Random Quantities, and Trievents", in *Philosophical Lectures on Probability*, ed. by Mura, A., trans. by Hosni, H., Vol. 340, Synthese Library, Dordrecht: Springer, 175-85, originally published as "La probabilità come prezzo", in Id. *Filosofia della probabilità*, ed. by Mura, A., Milano: Il Saggiatore, 253-62).

DeRose, K. and Grandy, R.E. (1999), "Conditional Assertions and "Biscuit" Conditionals", *Noûs*, 33, 405-20.

Edgington, D. (2007), "On Conditionals", in Gabbay, D.M. and Guenthner, F. (eds.), *Handbook of Philosophical Logic*, 2nd edition, Vol. 14, Dordrecht: Springer, 127-221.

Égré, P., Rossi, L. and Sprenger, J. (2021a), "De Finettian Logics of Indicative Conditionals – Part 1: Trivalent Semantics and Validity", *Journal of Philosophical Logic*, 50, 187-213.

Égré, P., Rossi, L. and Sprenger, J. (2021b), "De Finettian Logics of Indicative Conditionals – Part 2: Proof Theory and Algebraic Semantics", *Journal of Philosophical Logic*, 50, 215-47.

Gibbard, A. (1981), "Two Recent Theories of Conditionals", in Harper, W.L., Stalnaker, R. and Pearce, G. (eds.), *IFS: Conditionals, Belief, Decision, Chance, and Time*, Dordrecht: Reidel, 211-47.

Hájek, A. (1989), "Probability of Conditionals – Revisited", *Journal of Philosophical Logic*, 18, 423-28.

Kleene, S.C. (1938), "On Notation for Ordinal Numbers", *Journal of Symbolic Logic*, 3, 150-55.

Kleiter, G.D. (2018), "Adams' p-validity in the Research on Human Reasoning", *Journal of Applied Logics – IFCoLog Journal of Logics and their Applications*, 5, 775-825.

Kripke, S. (1975), "Outline of a Theory of Truth", *The Journal of Philosophy*, 72, 690-716.

Lassiter, D. and Baratgin, J. (2021), "Nested Conditionals and Genericity in the de Finetti Semantics", *Thought: A Journal of Philosophy*, 10, 42-52.

Lewis, D. (1973), *Counterfactuals*, Oxford: Basil Blackwell.

Lewis, D. (1976), "Probabilities of Conditionals and Conditional Probabilities", *Philosophical Review*, 85, 297-315.

Łukasiewicz, J. (1930) (1987), "Philosophical Remarks on Many-Valued Systems of Propositional Logic", in *Polish Logic: 1920-1939*, ed. by McCall, S., Oxford: Oxford University Press, 66-87, originally published as "Philosophische Bemerkungen zu mehrwertigen Systemen des Aussagenkalküls", *Comptes rendus de la Société des Sciences et des Lettres de Varsovie*, cl. iii, 23, 1930, 5177.

Lycan, W.G. (2001), *Real Conditionals*, Oxford: Oxford University Press.

Mackie, J. (1973), "The Logic of Conditionals", in *Truth, Probability and Paradox: Studies in Philosophical Logic*, Oxford: Oxford University Press, 109-19.

McGee, B. (1994), "Learning the Impossible", in Eells, E. and Skyrms, B. (eds.), *Probability and Conditionals*, Cambridge: Cambridge University Press, 179-99.

McGee, V. (1981), "Finite Matrices and the Logic of Conditionals", *Journal of Philosophical Logic*, 10, 349-51.

McGee, V. (1985), "A Counterexample to Modus Ponens", *The Journal of Philosophy*, 82, 462-71.

Milne, P. (1997), "Bruno de Finetti and the Logic of Conditional Events", *The British Journal for the Philosophy of Science*, 48, 195-232.

Milne, P. (2004), "Algebras of Intervals and a Logic of Conditional Assertions", *Journal of Philosophical Logic*, 33, 497-548.

Mura, A. (1994), "Fairness and Expectation", in Cellucci, C., Di Maio, M.C. and Roncaglia, G. (eds.), *Atti del Congresso* Logica e filosofia della scienza: Problemi e prospettive*, Lucca, 7-10 gennaio 1993*, Pisa: ETS, 455-74.

Mura, A. (2009), "Probability and the Logic of de Finetti's Trievents", in Galavotti, M. (ed.), *Bruno de Finetti Radical Probabilist*, London: College Publications, 201-42.

Mura, A. (2016), "Logica dei condizionali e logica della probabilità", *Rivista di filosofia*, 107, 71-98.

Quine, V.W.O. (1966), *Methods of Logic*, 2nd edition, New York: Holt, Rinehart and Winston.

Ramsey, F.P. (1928) (1990), "Law and Causality", in Mellor, D.H. (ed.), *Philosophical Papers*, Cambridge: Cambridge University Press, 140-63.

Schulz, M. (2009), "A Note on Two Theorems by Adams and McGee", *Review of Symbolic Logic*, 2, 509-16.

Stalnaker, R. (1970), "Probability and Conditionals", *Philosophy of Science*, 37, 64-80.

Stalnaker, R. and Thomason, R. (1970), "A Semantic Analysis of Conditional Logic", *Theoria*, 36, 23-42.

van Fraassen, B.C. (1966), "Singular Terms, Truth-Value Gaps, and Free Logic", *The Journal of Philosophy*, 63, 481-95.

Wright, G.H. von (1957), "On Conditionals", in *Logical Studies*, London: Routledge and Kegan Paul, 127-65.

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Book Reviews

Bongiovanni, Giorgio; Postema, Gerald; Rotolo, Antonino; Sartor, Giovanni; Valentini, Chiara; Walton, Douglas (eds.), *Handbook of Legal Reasoning and Argumentation*.
Dordrecht: Springer, 2018, pp. xxiii + 764.

The volume is composed of three parts. The first, "Basic Concepts for Legal Reasoning", addresses a number of topics that are preliminary to an understanding and discussion of reasoning and argumentation in legal contexts. The titles of the chapters collected in this part read like this: Reasons (and Reasons in Philosophy of Law); Reasons in Moral Philosophy; Legal Reasoning and Argumentation; Norms; Norms in Action; the Goals of Norms; Values; Authority; and the Authority of Law.

The second part of the book, "Kinds of Reasoning and the Law", explores the ways and the extent to which some general patterns of reasoning figure in legal practice and decision-making. These are the topics: Deductive and Deontic Reasoning; Inductive, Abductive, and Probabilistic Reasoning; Defeasibility in Law; Analogical Reasoning; Teleological Reasoning in Law; Interactive Decision-Making and Morality.

The third and last part of the volume, "Special Kinds of Legal Reasoning", focuses on some aspects and forms of reasoning that characterize legal practice and make it different from other argumentative practices in several respects. These are the topics addressed here: Evidential Reasoning; Interpretive Arguments and the Application of the Law; Statutory Interpretation as Argumentation; Varieties of Vagueness in the Law; Balancing, Proportionality and Constitutional Rights; a Quantitative Approach to Proportionality; Coherence and Systematization in Law; Precedent and Legal Analogy; Economic Logic and Legal Logic.

All contributors are renowned scholars and have made a valuable effort to put in concise and effective form a subject-matter which is multifaceted and somewhat unstable, in that it depends on the perspectives and vagaries of different legal systems and jurisdictions. All in all, the volume is a treasure trove of information and philosophical insights into legal reasoning and argumentation; it is and would be an excellent tool for those who want to learn about that, as well as for those who want to engage in scholarly debates.

An introduction by the late Douglas Walton precedes everything and sets the stage for the more detailed presentations that follow. In particular, in his own chapter, Walton makes it clear that argumentation is not the same as reasoning, because in an argument "the conclusion is always the claim made by one party that is doubted or is open to doubt by the other party. The other party may be a single person or an audience composed of more than one person, for example a jury. In argument, the conclusion is always unsettled, or open to doubt. Indeed, that is the whole point of using an argument. If there is no doubt about a proposition, and everybody accepts it as true, there is no reason for arguing either for or against it" (68). So, argumentation involves reasoning and has distinguishing features. Among these, an argument is performed in public; it is about a disputed point or claim; and it is part of a dialectical exchange, where critical questions are posed and the claim is unsettled.

The focus on reasons (rather than rules of positive law, or rules of other sorts) characterizes the first two chapters (by Giorgio Bongiovanni and Carla Bagnoli respectively) on reasons in philosophy of law and in moral philosophy. This

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partakes in the contemporary shifting of attention *from rules to reasons*.[1] The understanding that reasons guiding legal practice and argumentation are antecedent to, and consequently more important than, specific rules of positive law has opened up new paths of inquiry in legal scholarship and reflection. This volume contributes to this in a significant manner. Additionally, the primacy of reasons has created a sizeable common ground for theorizing on legal practice by scholars from different countries and traditions. It was once said, as a commonplace, that the legal world was divided into common law and civil law countries, and that their differences were remarkable. With a focus on reasons rather than rules or specific procedural arrangements, the gap between argumentation in the Anglo-American common law systems and the European civil law systems, which was once understood as dramatic, has become much smaller.

As I already said, there is a lot of material in this book. This is good, even if some parts are not strictly speaking necessary for an account of legal reasoning and argumentation (I wonder about the two chapters on authority,[2] and about some parts that, like the chapter on values, do not deal with legal issues directly), and even if some parts overlap to a certain degree (for example, the chapter on evidential reasoning and the one on inductive, abductive and probabilistic reasoning, or the chapter on interpretive arguments and the one on statutory interpretation as argumentation).[3]

Given space constraints and my limited expertise, I will just add a few comments on some specific topics. Remember first that argumentation is pervasive in legal practice: legislators argue about statutes to enact, parties in a dispute argue about their claims and counterclaims, judges and jurors argue about the arguments of the parties, other judges argue about the reasons given to justify earlier judicial decisions, and so on. Very roughly, legal arguments in a judicial context can be divided into *evidentiary* and *interpretive* ones. The former deal with the evidence presented to support a factual claim (e.g., in a civil case, the claim that the plaintiff was injured by the defendant's negligent driving, or, in criminal case, that the victim was shot by the defendant during a robbery attempt). The latter deal with the interpretive canons employed to extract normative content from legal provisions or texts (e.g. the argument from literal meaning, the argument from legislative intent, the argument from purpose, the argument from systemic coherence, etc.). As a challenging case for interpreters, consider for instance the well-known *Smith v. United States*, where the United States Supreme Court had to decide in 1993 whether the exchange of a gun for drugs constituted "use" of the firearm "during and in relation to" a drug trafficking crime, within the meaning of the relevant federal statute (see 576-578, in the chapter by Andrei Marmor). The Court said so and supported its decision, basically, with an argument from literal meaning ("use" means any use) and an argument from purpose ("drugs

[1] See also, e.g., the introduction to Dahlman, C., Stein, A. and Tuzet, G. (eds.) 2021, *Philosophical Foundations of Evidence Law*, Oxford: Oxford University Press.
[2] Of course it is important to see that authoritative sources are a constraint in legal reasoning and argumentation: in the application of law one cannot disregard what the relevant authorities (constitutional framers, legislators, judges creating precedents, administrative agencies and the like) established. I wonder whether this deserved two chapters.
[3] By the way, some overlapping is inevitable in works like this. For those who want to learn more on legal interpretation along argumentative lines, see Walton, D., Macagno, F. and Sartor, G. 2021, *Statutory Interpretation. Pragmatics and Argumentation*, Cambridge: Cambridge University Press.

and guns are a dangerous combination" and, by enacting the relevant statute, Congress wanted to minimize such risks). In a famous dissent, based on the use of language in context, Justice Scalia argued that to speak of "using a firearm" is to speak of using it for its distinctive purpose, i.e., as a weapon.

Quite reasonably, in their chapter on interpretive arguments and the application of law, Moreso and Chilovi say that "interpretive methods should not be conceived as separate elements of analysis: they should be considered as parts of an integrated method we use to determine legal content" (500).

When interpretive arguments are not sufficient because there is a gap in the law, one has to argue from *analogy*. Analogical reasoning fills the gaps in a legal system. As Bartosz Brożek points out in his chapter, the big problem lies in the assessment of the *relevant* similarities and differences between cases (368-370, 376-378). Everything is similar to everything else in some respect. The essential justification condition of an argument from analogy lies in the individuation of the relevant similarity. In the famous 1896 *Adams* case the alleged and conflicting similarities were two. This is how Brożek describes the case:

> Adams, a passenger on a boat operated by the New Jersey Steamboat Co., had some money stolen from his stateroom, despite having his door locked and windows fastened. The question the court had to answer was whether the defendant was liable as an insurer, i.e., without proof of negligence. There was no explicit rule stating the criteria for the responsibility of steamboat's operators. There were, however, other cases pertaining to the liability of service providers. In such cases as *Pinkerton v. Woodward* it was assumed that innkeepers were liable as insurers for their guests' losses. On the other hand, in cases such as *Carpenter v. N.Y.* … it was established that the operators of a berth in a sleeping car of a railroad company are liable only if negligent. There are analogies between steamboats and inns, as well as between steamboats and sleeping cars. The court considered both analogies and decided that the first one was more relevant, stating that the steamboat's operator is liable as an insurer (368).

Adams was a case of "dueling analogies",[4] for in a sense steamboats providing staterooms resemble inns, and in another sense as vehicles they resemble trains. The plaintiff argued for the first analogy, the defendant for the second. The court established the first as the relevant one, in order to protect the special trust relationship between the parties (provider of the service and customer). However, as a general point, for Brożek "there is no single, commonly accepted way of determining the relevant similarity between two cases" (378).

Finally, let me go back to evidentiary arguments. These are the arguments on matters of *evidence and proof.* They have a crucial importance for the outcome of a litigated case, since the application of the law is conditional on how the facts are reconstructed and categorized. Evidence is collected, admitted at trial, presented to the decision-makers, and evaluated to the purpose of reconstructing the relevant facts and make a correct decision on them. Evidentiary items become the content of arguments and evaluations by the parties first and the decision-makers then (e.g. the witness is reliable, the picture is ambiguous, the DNA evidence just shows that the defendant was there and not that he or she committed the crime, and so on). To transform evidence into proof one needs a "standard of proof".

[4] Schauer, F. 2009, *Thinking Like a Lawyer. A New Introduction to Legal Reasoning*, Cambridge, MA: Harvard University Press, 96-99.

Standard of proof (or burdens of proof, as they are also called) can be understood as decision thresholds.[5] Traditionally they come with qualitative formulations like "preponderance of the evidence" (civil standard) and "beyond reasonable doubt" (criminal standard). More recently scholars have been arguing about translating them into probability values and quantitative thresholds (> 0.50 for the civil standard, something like > 0.90 or more for the criminal standard). The shared assumption is that the criminal standard is more demanding in terms of evidential support because we generally think that convicting the innocent is worse than acquitting the guilty. To put it more technically, false positives (false convictions) are worse than false negatives (false acquittals). The extent to which it is so is a matter of debate. The more it is so, the higher must be the criminal threshold.

But the very translation of evidence into probabilities remains a controversial matter.

The chapter authored by Burkhard Schafer and Colin Aitken reviews the legal uses of inductive, abductive and probabilistic reasoning. Shafer and Aitken recall the philosophical and methodological issues involved in such uses and claim that, after having gone out of fashion due to Popperian falsificationism, induction has recently regained interest in the form of Bayesian confirmation theory (275). They distinguish different schools of Bayesian reasoning (objectivist and subjectivist) and point out that in legal cases statistical data are often not available, something which favors the subjectivist versions of Bayesianism (279).

The chapter authored by Marcello Di Bello and Bart Verheij presents three frameworks for the assessment of the evidence presented in a case, namely the argumentation framework, the probability framework and the scenario framework. In the first, one goes from evidence to arguments; in the second, from evidence to probabilities; in the third, evidence is assessed against scenarios. Di Bello and Verheij show the strengths and weaknesses of the three and reasonably prospect some integration between them (483).

DNA profiling and the use of Bayes' theorem are extensively discussed in both chapters. Bello and Verheij also mention Bayesian networks and ultimately highlight that no probabilities justify a decision by themselves: to this purpose we need standards of proof. A decision-theoretic framework must supplement any account of evidence assessment.

As a possible cross-fertilization between perspectives, to conclude, consider the issue of what I would call, in the absence of a better name, *argument ranking*. Let me focus on the arguments about the promotion of values and goals, and the arguments about the value of evidence. Sometimes these arguments are presented in quantitative terms (e.g. quantitative costs and benefits, probabilities) and objectors frequently claim that the relevant numbers are just arbitrarily chosen, for instance by attributing monetary values to legal goods in the context of proportionality analysis, or assigning arbitrary numbers to prior probabilities in the context of updating beliefs through Bayes' theorem. As Giovanni Sartor points out in his chapter on proportionality, "in most legal cases (at least when constitutional adjudication is at issue), we do not have sensible ways for assigning numbers and

[5] Cf. this characterization by Walton: "A burden of proof is a requirement set on one side or the other to meet a standard of proof in order for the argument of that side to be judged successful as a proof" (71).

constructing the corresponding functions" (614).[6] However, Sartor contends that we can "reason with non-numerical quantities" (615). Not only can we put things in ordinal rankings (e.g. this line is longer than that): we can also express non-numerical cardinal evaluations and quantitative proportions (e.g. this line is twice longer than that). This is a challenging and inspiring way to think about constitutional balancing and proportionality, if it is true that we can "compare situations where values are realised in different ways" (618). Similarly, we can compare evidentiary arguments along several dimensions and arrange them in rankings that are not only ordinal but also quantitative, provided that the conceptual link between quantitative and numerical is severed or at least diluted.[7] Some arguments are better than others, and we can say how much they are so if we can specify the relevant dimensions.

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Lapointe, Sandra (ed.), *Logic from Kant to Russell: Laying the Foundation for Analytic Philosophy*.
New York and London: Routledge 2019, pp. 255.

The edited volume *Logic from Kant to Russell: Laying the Foundations for Analytic Philosophy* aims to display the theoretical relevance of Kant's logic for the development of the 19^{th} and 20^{th} century logic. The volume does not belong to the philosophical genre of the histories of logic, but it collects a series of thematically organised essays. The chapters follow a chronological order and one may read them accordingly (25). This notwithstanding, they do not intend to provide a mere historical description, or rather a rational reconstruction of the long trajectories of thought that connect Kant and Russell, and that pave the way for the rising of analytic philosophy and the establishment of the paradigm of contemporary logic. None of the contributions exclusively assesses either Kant or Russell's views on logic, but they rather focus on famous and lesser-known thinkers that elaborated their views on logic in the time between the publication of Kant's *Critique of the Pure Reason* and Russell's *Principia Mathematica* (25). This alternative, non-mainstream narrative has Kant as its leading figure, and it intentionally leaves Frege in the background (*ibidem*). The volume also does not cover possible counter-narratives on a positive relation between Kant and a leading figure in this tradition as Frege, or on Kant and the analytic philosophy in general, provided by more theoretically shaped interpretations.[1] This methodological choice is indeed an explicit reaction against the mainstream narratives on the historical development of contemporary logic (1). In fact, the standard narrative depicts Kant both as adding

[6] On proportionality review and constitutional rights see also the chapter by Giorgio Bongiovanni and Chiara Valentini (581-612), providing also a valuable survey of the literature.
[7] This is in line with the *relative plausibility* approach on evidence and proof. See especially Allen, R. and Pardo, M. 2019, "Relative Plausibility and Its Critics", *The International Journal of Evidence and Proof*, 23, 5-59.
[1] Among these reconstructions, see the classic Sluga, H. 1980, *Gottlob Frege*, London: Routledge and Kegan Paul, especially on 58-61, and Reed, D. 2008, *Origins of the Analytical Philosophy*. *Kant and Frege*, London, New York: Continuum International Publishing Group. See also Hanna, R. 2001, *Kant and the Foundations of Analytic Philosophy*, Oxford: Oxford University Press, Heis J. 2014; "The Priority Principle from Kant to Frege", *Noûs*, 48, 2, 268-97.

an irrelevant contribution to the history of logic (2), and mostly as a widely criticised philosopher of mathematics (4). In addition, standard 20th century history of logic defines 19th century post-Kantian logic in terms of a "confused mixture of metaphysics and epistemology",[2] and similar narratives, as in Dummett's example (24), consider the publication of Frege's *Begriffsschrift* in 1879 as a revolutionary event in thought, which took place in complete independence from its context (*ibidem*). On the contrary, to justify its peculiar non-standard reconstructive approach focused both on "minor" figures in "Kant's wake" and on Kant's own conception of logic, the *ratio* of the volume is to put a great deal of attention on the methodology. In the Introduction, the editor Sandra Lapointe classifies six different alternative methodological approaches to the development of logic—rational reconstruction, contextualisation, doctrinal history, disciplinary history, thematic investigation and genealogy—and she states that the collective effort of the contributors intends to produce an example of "disciplinary history" (12). More precisely, the goal is to establish a disciplinary "historiography of logic" of the 19th century, rather than its history (5). By means of this definition, she identifies an interpretation of the development of logic backed by substantial theoretical and interpretive claims (5). With this respect, another key requirement for this approach is engaging with careful contextualisation (2), a feature that appears to be missing in the canonical texts in the history of logic that deal with Kant's conception of logic and its developments. In order to write a disciplinary history of logic, Lapointe deems necessary to look at the context in which different conceptions of logic in the 19th century have been put forward also by "minor" figures, avoiding thereby any retrospective judgement over their plausibility in light of modern standards, as it usually happens in rational reconstructions (10). On these premises, the interpretive assumption that lies at the core of the historiographical reconstruction is that Kant's "metaepistemological framework" (15) played a pivotal role in shaping the conception of logic in the 19th century. Said "metaepistemological framework" boils down to the distinction between pure general and transcendental logic, as well as between pure and applied general (i.e. psychology) and applied special (i.e. methodology) logic, drawn by Kant in Analytic of Concepts of the *Critique of the Pure Reason* (18). Choosing this very feature of Kant's logic as a guiding hypothesis for a historiographical reconstruction of 19th century logic is an insightful and fruitful move for several reasons. For instance, it fosters a more comprehensive reading of the evolution of the discipline and its "scopes and methods" (20) before (and beyond) its mathematisation, showing commonalities in the theoretical framework that are not reducible to mere historical continuities and do not resort to genealogies. As an additional result, the use of this metaepistemological framework for historiographical purposes proves to be in harmony with John MacFarlane's reading of the formality of Kant's logic. In fact, MacFarlane insisted on the significance of Kant's establishment of a pure general logic, characterising it as the first explicit theorisation of the "formality" of logic in its history.[3] By virtue of the inclusion of Kant's applied logic—conceived as

[2] I am referring to Kneale, M., Kneale W., 1962, *The Development of Logic*, London: Oxford University Press, 355: "For it was he [Kant] with his transcendentalism who began the production of the curious mixture of metaphysics and epistemology which was presented as logic by Hegel and the other idealists in the nineteenth century".
[3] See MacFarlane, J. 2000, *What Does It Mean to Say that Logic is Formal?*, University of Pittsburgh, PhD Dissertation, 95.

psychology and methodology[4]—in the metaepistemological framework, the volume collects contributions that enrich and deepen MacFarlane's proposal with respect to the historiography of logic of the 19th century. The chapters provide the validity of the claim with references both to notable logicians after Kant, such as Boole, Dedekind or Hilbert, and to lesser-known thinkers. In fact, in compliance with the method of disciplinary history, the volume targets a cluster of thinkers that Franz Ueberweg named "Logicians of the Kant's School" (20).[5] These logicians have elaborated on Kant's scarce remarks on pure general logic, highlighting further problems and issues that relate with parts of the metaepistemological framework mentioned above that go beyond general logic, such as methodology (18). Except for a couple of chapters, Kant's own doctrines and the relevance of the logicians of the School and their positions on logic are constantly under scrutiny in the volume.

Let us now look more closely at the contents of each chapter. In this overview I will try, when possible, to stress how Kant's logic enters in them. The first essay by Jeremy Heis epitomises the goals set in the Introduction, and thematises the Logicians of Kant's School between 1789 and 1851 by considering their relation to Kant. These thinkers, among the others Krug, Kiesewetter, Hamilton, Herbart and Mansel, emphasised a number of problems connected with Kant's division between thinking and knowing, which corresponds to the division between pure general and transcendental logic. Heis evaluates how the logicians of the school tackled problems such as the analyticity of formal logic, a precise determination of logical laws of thought, the formality of Kant's pure general logic, and Kant's theory on the formation of concepts. The second essay, by Graham Priest, and the third, by Clinton Tolley, deal with Hegel's logic. Priest traces back his renowned dialetheist interpretation of Hegel's logic to Kant's *Antinomies* in the *Critique of Pure Reason*, defining Hegel as the "zenith" of dialetheism in the history of philosophy between Aristotle and the present times (71), but highlighting at the same time the importance of the Kantian background in the formulation of his dialectic logic. Tolley focuses on Hegel's logic as well, and he engages in an attempt to rehabilitate the continuity of Hegel's conception of logic as "objective thinking" and "science of truth" (93) with the logicians in the century after Kant, downsizing in his reading its theological and hard-core metaphysical interpretations. Tolley does not refer to the logicians of the Kant's School, but rather to three different conceptions of logic (95), namely the mathematical-objectivist, semantical objectivist, and pragmatist-intersubjective conceptions, which are ascribable respectively to Russell, Frege and Brandom among the others. In the fourth essay, the editor of the volume Sandra Lapointe underlines the innovative views on logic put forward by the Czech mathematician and philosopher Bolzano. In doing so, she puts at work the methodological principles enucleated in the Introduction to the volume, and she argues that one may interpret Bolzano's logical reform through a different narrative in continuity with Kant (104-105). The standard narrative credits Bolzano with innovative theories on "antipsychologism and semantical realism, logical consequence and logical truth as invariance" (104). Lapointe argues that one shall not refer to a contemporary

[4] It shall be noted that with 'methodology' I hint at Kant's 'applied special logic'. Therefore, the term should be taken in a restricted sense.

[5] MacFarlane sketched also a description of the effects of Kant's 'discovery' of the formal character of logic on logicians in Germany and in Britain, before dealing in the dissertation with the very same theoretical issue in Frege. See MacFarlane, J. 2000, *What Does It Mean to Say that Logic is Formal?*, University of Pittsburgh, PhD Dissertation, 127-33.

account of analyticity (107) and of logical consequence (109) to account for the origin of Bolzano's theoretical novelties on these topics, but one should rather look to the logic of his time.

In the fifth essay, Lydia Patton addresses George Boole and his ground-breaking contribution to the theory on the relations between logic and algebra, to the point that he can be judged a "precursor to the model-theoretic approach" (123). She reconstructs how Krug, Esser and especially the debates on the status of logic among British post-Kantians, for instance Thomson, influenced Boole's account of logic (123-4). The result is surprising, and shows how Boole's application of logics to algebra is intertwined with more general theoretical problems, such as the question on the scientific status of logic and the objectiveness of its laws, that stem from Locke and Kant and from the subsequent debates in the "New Analytic" approach to logic (128-34). The sixth essay by Nicholas Stang defends a new interpretation of Lotze's logic, according to which he supported a form of "non-hypostatic Platonism" (139). Lotze had defended the existence of propositions, while at the same time denying the claim that they belong to a separate ontological realm. After a *detour* through the different senses in which Lotze's characterises the elements that are part of his ontology (141-147), Stang criticises a reading of Lotze that would ascribe a strong ontological conception of propositions to him, and he shows to what extent his alternative line of thought resembles the non-hypostatic reading of Frege's Platonism (151). Stang's characterisation of B-Platonism, that he does not attribute directly to Frege, is a useful tool to express Lotze's idea that the propositional content of judgement is objectively valid but not actual (*wirklich*), since it is neither spatially nor temporally extended, and not subject to causal laws. Hence, one may conceive of the objective content of judgements as integrally established by the laws of logic (157). In the seventh essay, Frederick Beiser exposes the late theory of logic of the Neo-Kantian Hermann Cohen. In the *Logik der reinen Erkenntnissen*, Cohen theorises a notion of "pure thinking" that has a strong idealistic flavour, given that it states that thought can produce its object *a priori*. Arguing against the hypothesis that Cohen may have given up on Neo-Kantianism, Beiser holds that Cohen is making reference to Kant's notion of "*a priori* thinking" in establishing its account of pure thinking (164). In the second part of the essay, Beiser goes through Cohen's theory of infinitesimals (166 and f.), and demonstrates how this notion plays a fundamental role in the explanation of qualitative and quantitative features of reality, being part of the "nomological idealism" Cohen would defend in his mature works (170). The eighth essay by Erich Reck is devoted to Dedekind and to his peculiar version of the logicist thesis. Roughly speaking, this thesis contrasts Kant's intuition-based doctrines on the discipline, and it states both that arithmetic is a part of logic (172) and that numbers are to be conceived set-theoretically. In contraposition to Kant, Dedekind explains space and time through the doctrine of real numbers, rather than the other way around. Despite the opposition to Kant's account of mathematics, and the fact that Dedekind shares fundamental innovative ideas with the logicists and with Frege (185), Reck argues that Dedekind thinks of logic in agreement with his time. His notion of thinking and his basic claims on the laws of thought were "pointing towards Kant's categories of the understanding" (183), whereas Frege thought of logic in a different way. Also in light of this, Reck acknowledges and defends the originality of Dedekind's version of logicism. The ninth, tenth and eleventh essays focus on Russell. In the ninth essay, Consuelo Preti provides a detailed explanation for an apparently perplexing statement Russell made in a letter to Couturat in 1900, in which he defined Moore as "the most subtle in pure logic" (190). The motivations behind this definition can be detected in Moore's innovations with respect to his philosophical background, in particular concerning his realist stance on the metaphysics of judgements. Preti takes into account how prominent figures such as Bradley and Kant contributed to the development of Moore's realist stance on logic, which had in turn an influence on Russell. The conception of psychology that was widespread in Cambridge at the end of the 19^{th} century plays an important role as well in this narrative: following the attempt to establish a "scientific psychology", Kant was read in Cambridge as a "bad psychologist" (196), and his theories of the morals were interpreted along the same lines. Moore's reaction to Kant's theories on morals exposed in the dissertation he delivered for his Trinity Fellowship in Cambridge, together with his rejection of Bradley's Idealism (202), are then crucial, according to Preti, to account for the positions that led to Russell's positive assessment of Moore as a logician. In the tenth essay, Sean Morris suggests a continuity between Russell's "idealistic" period and his works on epistemology and theory of knowledge in the 1910s, under the common concern on the methodology of scientific philosophy (206). A key step for proving this claim is to look at the German philosopher Sigwart, whom Russell praised and showed appreciation for (208). In his *Logik* (210 and f.), Sigwart underlines the importance of logic for scientific methodology concerning the logical perfection of judgements and the striving to systematicity of knowledge both with regards to its complete deductive derivability from principles and with regards to "*systematic classification*" (215 and f.). For Sigwart, both logic and empirical results should jointly contribute to the construction of a metaphysics that is contiguous to science. Morris observes the similarities between this conception and Russell's late theorisation of a scientific philosophy (226) that does not imply a foundationalist account of knowledge, but which conceives of philosophy as "complementary" to empirical science (232). According to Morris, this proves that the influence of Sigwart on Russell spans beyond the foundations of mathematics. The very last chapter by Nicholas Griffin analyses the presence of Kant's in the epigraph of Hilbert's *Foundations of Geometry* (235) and in the axiomatization of space given by Russell before embracing the logicist view (239). As for Hilbert, Griffin shows how in the lectures before 1898 the foundations of geometry were characterised by an appeal to the notion of space conceived along Kantian lines, i.e. regarding its perception in experience and in intuitions (237), although in the published book Kant is only quoted in the epigraph. As for Russell, Griffin argues that, even after he rejected Kantian transcendental arguments in geometry in 1897 (239), he remained under the influence of a Kantian conception of space in laying the foundations of geometry in algebraic terms before his logicist turn. This was due to the influence of Whitehead's "abstract's general idea of space", which underlies Russell's own conception of algebra. In light of this, Griffin proves that Russell's, as well as Hilbert's, initial axiomatizations of geometry were "glued" by "the faint reflection of Kant's 'fading glow'" (245).

To conclude, the volume is faithful to the methodological principles outlined in the Introduction and it proves worth reading. While it may appear not homogenous at first glance, the volume shows how an alternative narrative based on Kant can be both justified on solid methodological grounds and successfully applied to specific instances in the history of logic of the 19^{th} century.

University of Padua DAVIDE DALLA ROSA

Westphal, Kenneth, *Kant's Critical Epistemology*: *Why Epistemology Must Consider Judgment First*.
Abingdon: Routledge, 2020, pp. xxv + 369.

Westphal's project seeks to read Kant's *Critique of Pure Reason* against the transcendental idealist grain whilst highlighting resources and insights from Kant's commonsense perceptual realism. The book is divided into three parts: I) Epistemological Context, II) Kant's Critical Epistemology, and III) Further Ramifications. While Westphal commits to an impressive and sundry review of Kant's First Critique, balanced with Neo-Kantian bricolage, the central theses that he offers draw from Kant's three Analogies of Experience and the four Paralogisms of Rational Psychology, with interest in the relationship between Kant's theory of perceptual judgment and account of empirical knowledge.[1] Westphal makes the case that Kant's first Critique correctly defends a robust fallibilist account of empirical justification, an insight that has eclipsed most, if not all, previous Kantian interlocutors. Despite Westphal's book is brimming with analyses and critiques of philosophers inspired by and reacting to Kant, historical and contemporary, the true merits of Westphal's project are in his erudite parsing of the first Critique with cognitive semantics in mind.

The first three chapters, which comprise the first section, find Westphal situating Kant within the history of analytic epistemology. In developing this section, Westphal enumerates the state of epistemology prior to Kant—guided by the Cartesian assumption and epitomized by Hume, epistemology was anchored to evidential data, with states of sensory-consciousness undifferentiated from states of self-consciousness awareness. This presumption, when conjoined with infallibilist assumptions about cognitive justification—the infallibilist doctrine being that nothing short of provability suffices for justification—inevitably leads to the egocentric predicament of Cartesian skepticism and internalist infallibilism. Westphal's project stakes to evince that Kant is the first great non-Cartesian epistemologist, developing forms of externalism not only about mental content and causal judgment, but also about cognitive justification (49).

By the end of the first chapter, we see Westphal's thesis begin to take shape: that, although necessary, sensory stimulation is insufficient for cognitive warrant. Sense-data is such that we can process it by bringing it under concepts in judgments whereby we classify and identify the various particulars (objects, events, structures, processes or persons) surrounding us. Throughout Westphal's project, this will reappear in different applicatory scenarios, ranging from semantics to perceptual psychology to metaethics. Westphal's ultimate Critical endeavor is to poise Kant via scientific realism's mold, making the case that Kant's anti-skeptical transcendental proof(s) demonstrates that any human being who is apperceptive—insofar as they are aware of some appearances appearing to occur before, during, or after others—"must actually perceive at least some particulars in her or his surroundings, in order to identify even a presumptive, approximate temporal sequence amongst appearances" (219). Situating Kantian epistemology historically throughout these first three chapters, Westphal cites a number of contemporary epistemological puzzles, such as Gettier-type problems regarding justified true belief and the examples

[1] Kant, I. 1998, *Critique of Pure Reason*, Cambridge: Cambridge University Press, §A190, 192– 3/B235, 237-38, 275.

therewith, which centrally involve what are termed "externalist" factors bearing upon the justificatory status of Someone's beliefs—factors such that Someone cannot become aware of truth-laden belief(s) by simple reflection. Following Descartes' cogito argument, stilted on the putative self-transparency of beliefs qua ideas—and those Cartesian epistemologists prioritizing access-internalist infallibilism regarding inner experience—"internalism" was launched in the service of what Westphal terms "global perceptual skepticism".[2] Kant's fallibilism, and his transcendental proof that we can only be self-conscious of our existence as determined in time via apperception if we have *some* perceptual experience and knowledge of spatio-temporal causally active substances in our surroundings, counters the skeptical generalization from *occasional* perceptual error to the possibility of universal perceptual error (or, *mutatis mutandis*, insufficient cognitive justification):

> [...] any world in which we are altogether perceptually deluded is a world in which no human being can be apperceptive [...]. Global perceptual sceptics simply assume that we can be self-conscious without being conscious of anything outside our minds. Kant's transcendental proof of realism shows just how portentous is this assumption (227-28).

Furthermore, Kant's three principles of causal judgment, as detailed in the three Analogies of Experience anchor Westphal's description of our cognitive capacity for identifying enduring events:

1. Substance persists through changes of state.
2. Changes of state in any one substance are regular or law governed.
3. Causal relations between substances are causal interactions (147).

Kant's three Analogies are universally quantified and these principles *guide* causal judgment. Moving from phenomenal causality to cognitive semantics, having now broadly outlined his project's ambitions, Westphal's second section, "Kant's Critical Epistemology", is comprised of six chapters (viz., chapters 4-9). Notably, it is in the fourth chapter, "Constructing Kant's Critique of Pure Reason", where Westphal begins to formalize Kant's semantics of singular, specifically cognitive, reference, prodding philosophy of language, epistemology, and Kant scholarship into truly novel and exciting territory. Westphal first makes the general case that to understand empirical knowledge we must distinguish between predication as a grammatical form of sentences, statements or (candidate) judgments, and predication as a (proto-)cognitive act of ascribing some characteristic(s) or feature(s) to some localized particular(s). By way of Kant, Westphal argues that term "particulars" ought to be construed broadly so as to include any kind of particular we may localize within space and time. Kant sought to expound upon a general phenomenon rather than individual facts, thus systematizing how natural regularities can be and are localized. Westphal argues that Kant's semantics of singular reference achieves verification empiricism *without* invoking empiricism. Contra verificationist theories of meaning—which only require logically consistent propositions—and whether stated in terms of concepts, propositions, or judgments, Kant's justification of realism involves explicating classificatory content

[2] Descartes, R. 1985. *The Philosophical Writings of Descartes*, 3 Vols., J. Cottingham, R. Stoothoff, D. Murdoch, A. Kenny (eds. & trs.), Cambridge: Cambridge University Press.

descriptions *vis-à-vis* further requirements involved in *actually* classifying or identifying any extant instance so-described, doing so accurately, warrantedly/justifiedly and, thus, cognizantly.

By the sixth chapter, Westphal has successfully bridged Kant's objective significance and justifiable cognitive judgment with the refutation of global perceptual skepticism. Thus follows one of Westphal's most interesting developments: drawing from the Transcendental Deduction's description of synthesis in apprehension, where perception must fully accord with the category of quantity, Kant's Thesis of Singular Cognitive Reference "concerns the cognitive, and hence also the epistemological significance of identifying by locating those individuals to which we ascribe any features, by which alone we can know them and can claim to have knowledge of them" (117). Constructing Kant's Semantics of Singular Cognitive Reference, Westphal espouses Gareth Evans' notion of predication as ascription, which requires conjointly specifying a relevant spatio-temporal region and manifest characteristics of any particular that we self-consciously experience or identify (§55).[3] These conjoint specifications allow for the ascription of manifest characteristics that are mutually independent cognitive achievements, integrating sensation/sense-data and conception/understanding through co-operation and integration. Westphal eventually develops a Critical method wherein:

> Sensibility is required (though not sufficient) for sensing the various manifest characteristics of the sensed particular, and directing us to its location; Understanding is required (though not sufficient) for explicitly identifying its region and its manifest characteristics, thus enabling us to be apperceptively aware of this particular (262).

Westphal argues that Kant's Thesis of Singular Cognitive Reference services epistemology by substantiating that knowledge, justified belief, or experience of or about particulars require satisfying further conditions than those of conceptual content ("intension") or linguistic meaning alone. No matter how specified or detailed a description/intension may be, it cannot, by itself, determine whether it is referentially empty, determinate, or ambiguous because it describes *what* there is: either zero, one, or several individuals. However, to *know* any spatio-temporal particular requires correctly ascribing *characteristics* to it and localizing it in space and time. Via ostensive designation, we ascribe predicates used in our judgments to some putatively known particular, differentiating and characterizing it. The ascription of characteristics is required for singular, specifically *cognitive*, reference to a spatio-temporal particular, providing the necessary requirement for the truth-evaluability of our claims.

Between Chapters 6-9, Westphal aims to further enrich Kant's cognitive semantics qua particulars in order to provide a legitimate stand-alone epistemological doctrine. It follows that, insofar as epistemological "success term(s)" are considered, logical consistency requires that Someone *uses* that predicative proposition ascriptively to describe characteristics or features to some localized particulars. Kant's transcendental sense of "real possibility" denies that descriptions alone suffice for knowledge—no description suffices to specify and therefore determine whether there is any particular in some specific context by way of sentential meaning, as reference to some extant perceptual particular is required. Westphal pellucidly

[3] Evans, G. 1975, "Identity and Predication", *Journal of Philosophy*, 72, 13, 343-63.

writes that "only when the performer known as Prince ordered and purchased a flamboyantly purple guitar did the concept 'purple guitar' come to have 'real possibility' in Kant's full, referable-in-practice, empirical sense of this designation" (246). This undermines Russellian-cum-Quinean confidence in mere intension (predicates as classifications, explicated as mere descriptive phrases) and regimenting indexicals. Kant's demonstrative ("deictic") reference is required to obtain even *candidate* cognitive claims. Speaking does not suffice to speak *about* any individual thing, person, event, structure. Merely speaking or thinking intelligibly/understandably requires avoiding self-contradiction, whereas cognition or any claim to knowledge requires localizing the putatively known individual(s) within space and time, together with some approximately correct attribution of characteristics to it or them. Only in referential contexts can we advance from uttering sentences to making any epistemically warranted cognitive statement or claim (§89).

More broadly, Westphal's point is that empirical knowledge and semantic meaning involve more than simply supplying values for logical variables, as such stipulations, by design, abstract from descriptive identification and intelligibility while presuming purported reference. Reading Kant's reference-in-practice *vis-à-vis* Tetens' *realisieren*, Westphal articulates a key "deictic point" central to the conditions that must be satisfied so as to be able to make any sufficiently accurate attribution to even *claim* that something is such-and-so:

> S/he must localize that (or those) particulars to which (or to whom) S/he purports to ascribe any feature(s), so as (putatively) to know (cognize) it or them. Cognition is not secured by fortunate guesses in the form of mere descriptions which happen to have (had) some instance somewhere or other within nature or history. Cognition requires identifying by locating relevant particulars so as to be able to know them, or even to mistake them! (118)

Truth pertaining to knowledge, and therefore to epistemology, requires demonstrative reference to relative particulars. Only under these conditions can there by candidate objects of knowledge. Westphal's project recalls Carnap's "descriptive semantics"—the pragmatic use of propositions when making cognitive judgments in suitable perceptual or experimental contexts about localized individuals/particulars.[4] As demonstrated by Kant's Analogies, the causal principles regulating our causal judgments do so by guiding our identifying efficient causes of observed spatio-temporal events. Making such discriminatory, perceptual-causal judgments to identify particulars within our surroundings requires anticipation and modal imagination to consider relevant causally possible alternatives to the apparently perceived causal scenario. Westphal here argues that Kant's conception of "imagination" is not simply imaging/picture-thinking, but empirically informed counterfactual reasoning about causal possibilities.

The constitutive point in Kant's three Analogies involves our typically reliable capacities to distinguish and discriminate various kinds of causal sequences and processes amongst the perceptible, causally structured, and interacting particulars that surround us (§§48-49).[5] These particulars regulate our causal judgments. Were we unable to make any such causal discrimination(s) and identification(s) accurately and justifiedly, we would altogether lack apperception of our

[4] Carnap, R. 1956, *Meaning and Necessity*, Chicago: University of Chicago Press.
[5] Kant, I. *Critique of Pure Reason*, §A84-130/B116-69.

own existence as determined in time. Westphal's cognitive-semantic point here has far-reaching relevance for philosophy of language and epistemology, as well as for the history and philosophy of science, theory of action, and philosophy of mind. As will become the nexus for the third section of Westphal's book, which concerns scientific realism, Kant's cognitive semantics is embedded in and strongly supports Newton's causal realism regarding gravitational force—Westphal makes the case that Newton's methodological Rule 4 of (experimental) philosophy requires any competing scientific hypothesis to have not merely empirical evidence in its favor but also sufficient evidence with sufficient precision to either make an accepted scientific theory or law more exact or to restrict it and demonstrate exceptions to the rule (§§66-67). It is here that Westphal's reading of Kant, rigorous and unique when applied to semantics and epistemology, feels somewhat wanting—while the reader will feel assured that Kant's context-bound externalist epistemology warrants cognitive application within the non-formal domain of empirical knowledge, the diachronic development of physics and other such natural sciences are necessarily tethered to the uptake of particulars (i.e., replicated experiments and tests). Indeed, the Sellarsian apothegm rings true that there are as many scientific images of man as there are sciences which have something to say about man, where each science deploys distinct instruments and methods. It would thus be fruitful if Westphal, particularly given his Hegelian expertise, further explored the always-developing and self-correcting descriptive and explanatory resources of the scientific image and how it shapes rational judgments, which cannot be exhausted by the causal locutions of justificatory judgment, while at once pointing towards a radically non-normative picture of ourselves. Westphal briefly touches on this important consideration but his elaboration of Kant's work on transeunt causal action via rule-governed succession of states does not contend with the irresolvable frame-bound discrepancies between various scientific theories (quantum mechanics vs. Newtonians classical mechanics) or quantum measurement (viz. perceptual observation overdetermines superposition).

Despite this very minor limitation, Westphal's engagement with Kant *vis-à-vis* the history of philosophy is extremely fertile. The second section's latter chapters find Westphal reviewing Kant's inventory of cognitive capacities, describing Kant's insights into rational judgment as articulating "sensationism" about sensations, the view that sensations typically are components of acts of awareness of particulars. Situating Kant as steeped in the Humean predicament of psychological epistemology, Westphal illuminates Kant's account of consciousness by parsing an issue pertinent to contemporary representationalist accounts of perception—that if a sensory idea is caused by an object, then that idea also represents some feature of that object. In the philosophy of perception and neurophysiology, this issue transpires in the "binding problem(s)"—a problem concerning cognitive psychology that deals with explaining what unites any group of sensations into what might be a unified, fluid percept of any one object (§22). This problem arises synchronically within any moment of perception of an object and arises diachronically as a problem of integrating successive percepts of the same object: one set of issues is sensory, concerning the generation of sensory appearances to each of us; the second set is intellectual, concerning how we recognize the various parcels of sensory information we receive through sensory experience to be information about a spatio-temporally consistent object. Westphal makes the case that Kant's Transcendental Logic may provide us with a helpful conceptual primer here, as it concerns the kinds of judgment (classification, differentiation, conditionalization) required to identify, distinguish, track and classify individuals perceived in our surroundings. Although Westphal is not the first philosopher to cull Kant's unity of consciousness as relevant to the binding problem, the case study strengthens Westphal judgment-first epistemological approach, with the *a priori* concepts space and time utilized to identify any (actual) region of space and period of time in which various particulars change, are perceived, and are arranged.

Chapter 8 and 9 are perhaps Westphal's strongest chapters. It is here that Kant's epistemological findings about perception and causal judgment crystalize, with Westphal elaborating on Kant's proofs of content externalism. It follows that any world in which human beings are capable of apperceptive experience is one that must provide us some minimal regularity and variety amongst the contents of our sensations. This is what allows us to make judgments by way of identifying objects or events, for it is by way of judgment, and not sense-data, that we can distinguish ourselves from the objects that populate our environs and achieve apperception (§51). Kant's semantic point about singular cognitive reference and the proof of mental content externalism are here reinforced by his proof that we can only make legitimate causal judgments about spatio-temporal particulars (viz., persisting substances) using our conceptual categories.

The final third of the book, titled Further Ramifications, comprises four chapters. Chapter 10 elaborates on the aforementioned thesis regarding scientific realism, which veers towards a programmatic Carnapian rendering. However, it is Westphal's consideration of the free will vs. determinism debate that occupies the bulk of the final chapters. Westphal approaches this debate qua metaphysics rather than metaethics and, as is characteristic of Westphal's reading—contra those interpreters who contend that Kant's compatibilism entails the truth of causal determinism and, thus, insist upon the wellspring of the noumenon for radical freedom—Westphal reads Kant's argument here without appealing to his transcendental idealism. Westphal argues that Kant reveals the entire free will vs. determinism debate as void, intractable, and an *argumentum ad ignorantium* (§§74-83). This will undoubtedly serve as the most controversial section for those Kant scholars who uphold the "two-worlds" view as key to linking Kant's practical and theoretical philosophy. Nevertheless, Westphal's judgment-first approach offers a robust conception of normativity, where "rational judgment is normatively structured insofar as it consists in critical assessment of justifying grounds, principles, evidence and our use of them in any specific judgment, and because the normative character of justificatory judgment cannot be reduced to, nor eliminated by, causal considerations" (288). In Chapters 11-12, Westphal argues that Kant's account of causal judgment suffices to preserve the possibility of free and imputable action at the psychological level. Westphal underscores that we can *only* make accurate and justifiable causal judgments about spatio-temporal particulars—causal knowledge results from successful, exclusively causal explanation of actual events but the principle of universal causal determinism is not, nor can be, a known causal law at the psychological register.

Reviewing Kant's Paralogisms of Pure Reason, Westphal asserts that we have well-justified causal beliefs *only* to the extent that we have credible evidence for causal explanation of events.[6] Consequently, the transcendental causal principle, that every event has a cause, is a *regulative* principle of causal inquiry and we

[6] Ibid., A341–61, B399–413.

obtain causal knowledge only from successful causal explanation, which does not obtain for inner psychology (mental events). Mistaking the causal principle for a justified causal law is an instance of "transcendental subreption", of "mistaking conditions for the possibility of human experience for substantive features of the world we experience" (167). Westphal emphasizes that Kant's principles of causal judgment, as justified in the "Analogies", only hold when referred to spatio-temporal substances; via *modus tollens*, causal judgment *cannot* be known to hold of merely psychological phenomena (§§45-46). Here, the physicalist may rejoinder: but inner psychology is composed of physical neural events, and thus spatio-temporal particulars-cum-substances which we can represent and use as positive empirical evidence given our contemporary brain-imaging techniques (e.g., fMRI, EKG)! Westphal does not consider such responses, leaving (naturalist) readers who might agree with Westphal's sidelining Kant's transcendental idealism teeming with such queries. Nevertheless, Westphal also takes a second approach to determinism, not via psychology but bodily behavior. Appealing to Kant's transcendental justification of bodily comportment within perceptible judgment, Westphal claims that causal behavior is underinformed and that identifying causally interacting substances in our surroundings does not justify causal determinism universally across the domain of spatio-temporal events. Westphal links his conception of the freedom of behavior to the semantics of cognitive behavior via the Principle of Sufficient Reason (PSR)—that every event has a sufficient cause or causes—claiming that it suffices as a regulative principle guiding causal inquiry, causal explanation, and causal judgment; it is not, nor can it be, a principle known to hold constitutively of all events within space and time (§§79-80). Westphal warns against our mistaking the PSR for an "unrestricted universal, demonstrated (i.e., cognitively fully and unrestrictedly justified) assertoric *law of causality*" (299)—we must never mistake a *principle* of causal inquiry for successful *outcomes* of such inquiry.

Westphal underscores that Richard McCarty conflates the causal principle—that each spatio-temporal event has (a) numerically distinct spatio-temporal cause(s)—for an established assertoric causal law, whereby every event in fact has some sufficient set of causes.[7] Westphal responds that: "Kant's Critical strictures on causal judgments within the merely temporal psychological domain entail that we cannot know pro or contra whether psychological phenomena are causally structured, or are causally deterministic" (321). Westphal is correct that a complete cause-and-effect schema will, necessarily, always be incomplete: enumerating a causally-closed map will forever be undermined by the nature of open systems, i.e., the fact that space and time are always present. But does this preclude reflection on causal determinism via best-inference? For Westphal, in the domain of human behavior, such attempts will make use of unjustified suppositions based on under-informed models, which are supplanted by highly abbreviated and short-hand causal commands.

Rather than relaying his critique to develop a metaethical doctrine separate from the Categorical Imperative and its noumenal purview, Westphal's methodological concerns brings him to conclude the book by advocating scientific realism. This will, indeed, satisfy naturalist Kantians like myself who are favorable towards Sellars' rendering. For Westphal, the supposition that mere logical possibilities undermine cognitive justification remains pervasive and props up

[7] McCarty, R. 2009, *Kant's Theory of Action*, Oxford: Oxford University Press.

cognitive skepticism, tendering multitudinous concerns in epistemology and philosophy of mind like the "hard problem" of consciousness, which trade in logical possibilities rather than demonstrative reference. Westphal's critique is leveled at philosophical methodologies that ascribe various characteristics to something that does not suffice for any actual ascription—as delineated by the semantics of cognitive reference, actual ascription always requires localizing relevant particular(s) sufficiently to discriminate them.

For Westphal, the conjoint implication of the Analogies of Experience and the Paralogisms of Rational Psychology is that we cannot make any legitimate, justifiable causal judgments about internal, psychological, or temporal states/occurrences. While there may loom large the impulse to project the universal determinist principle of binding causality from the constitutive principle of objective experience, Westphal is quick to remind us that this is what Kant criticized as "transcendental subreption"—mistaking transcendental conditions of the possibility of apperceptive human experience and knowledge for ontological conditions constitutive of spatio-temporal objects. Westphal conclusively claims that the debate of determinism vs. free will is not only deeply unsatisfactory but an empty question; philosophy would do better to engage in exercises of specific judgment or matters of action via the compatibilist framework that asks "[t]o what extent, or in what regard(s) is each action free?" (304). One hopes, however, that Westphal is not content with deeming the entire Kantian-metaethical purview of practical philosophy an altogether empty pursuit—it is here that the reader may underscore that the determinism vs. free will debate is tethered to critical questions concerning responsibility, culpability, and freedom. This debate informs our evaluative norms, reactive attitudes, and pragmatics, down to influencing jurisprudence and legislation; opting out of the debate may not be a choice when so much of our moral system is carved around it. Considering that reasons for doing are never categorially given, like sense-data, and that no moral particulars can be identified *a priori*, Westphal's prescription risks lapsing into abdication. Although Westphal is not a moral philosopher, having stepped into the metaethical boxing ring, the onus looms large for Westphal, and us as his readers, to grapple with how, and if, a judgment-first epistemology obtains in the metaethical terrain. Despite this query—which Westphal's construction very well may provide an answer to, although it must be made explicit—Westphal's epistemological rendering of Kant, particularly his work on cognitive semantics and content externalism, achieves the goal of proving Kant a meticulous epistemologist.

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