Is the de Finetti Conditional a Conditional?

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Abstract

In this paper, we apply a Herzberger-style semantics to deal with the question: is the de Finetti conditional a conditional? The question is pressing, in view of the inferential behavior of the de Finetti conditional: it allows for inferences that seem quite unexpected for a conditional. The semantics we advance here for the de Finetti conditional is simply the classical semantics for material conditional, with a further dimension whose understanding depends on the kind of application one has in mind. We discuss such possible applications and how they cover ground already advanced in the literature.

Keywords: de Finetti conditional, Material conditional, Herzberger semantics, Classical logic.

1. Introduction

If one is looking for controversy, then, one just needs to check debates on the meaning of conditionals. A quite intuitive idea behind it concerns the claim that asserting an indicative conditional $A \to C$ is not the same as making a single assertion. Rather, one makes a conditional assertion: it amounts to assuming the antecedent A and, based on that, reason on whether or not to assert C. This kind of approach leads one to the following evaluation of a conditional: assuming that A is the case, the conditional will be true in case C is true, and the conditional will be false in case C is false.

Now, the major intuition running behind such a proposal also suggests that whenever the antecedent A does not eventuate, there is no point in asking for the consequent, given that the circumstances allowing for an evaluation of the conditional are not fulfilled; we simply have no grounds to evaluate the proposition appearing as consequent. Bruno de Finetti advanced such an understanding of the conditional, as based on what he called 'trievents':

> A and B being any two events (propositions) whatever, we will speak of the *trievent* A/B (A given B), the logical entity which is considered: 1) *true* if A and B are true;

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> 2) *false* if A is false and B true; 3) *null* if B is false (one does not distinguish between "not B and A" and "not B and not A", the trievent being only a function of B and $(A.B)$ (de Finetti 1995: 184).

So, it results that $A \to C$ is true in case $A \wedge C$ is true (with '$\wedge$' being the conjunction) and is false in case $A \wedge \neg C$ is true; the conditional results simply *not determined* in other cases. We now present the idea in more details.[1]

Let the language $\mathcal{L}$ consist of the set $\{\neg, \to\}$ of propositional connectives and a countable set *Prop* of propositional variables which we denote by p, q etc.[2] Furthermore, we denote by *Form* the set of formulas defined as usual in $\mathcal{L}$. We denote a formula of $\mathcal{L}$ by A, B, C, etc. and a set of formulas of $\mathcal{L}$ by Γ, Δ, Σ, etc.

Then, assuming that a proposition may be true, false, or undetermined, as explained in the previous informal account, de Finetti's truth tables are as follows.

Definition 1. A *de Finetti interpretation* for the language $\mathcal{L}$ is a function I: *Prop* $\longrightarrow \{\mathbf{t}, \mathbf{u}, \mathbf{f}\}$. Interpretations I are then extended to valuations v by the following truth table.

	$\neg$
t	**f**
u	**u**
f	**t**

$\to$	**t**	**u**	**f**
t	**t**	**u**	**f**
u	**u**	**u**	**u**
f	**u**	**u**	**u**

Based on this, the simplest way to define a consequence relation will be in terms of the preservation of the value **t**.

Definition 2. For all $\Gamma \cup \{A\} \subseteq$ *Form*, $\Gamma \vDash_{dF} A$ iff for all de Finetti interpretations I, $v(A) = \mathbf{t}$ if $v(B) = \mathbf{t}$, for all $B \in \Gamma$.

Now, as observed in Egré, Rossi and Sprenger 2021: §3.2, we have the following consequences.

Proposition 3. The following holds for $\vDash_{dF}$.

(1) $A \to B \vDash_{dF} A$
(2) $A \to B \vDash_{dF} B$
(3) $A \to B \vDash_{dF} B \to A$
(4) $\nvDash_{\mathrm{dF}} p \to p$

So, despite the initial promising reading of the truth conditions for the conditional, what results, when it comes to the inferences that result valid, is that we have features that are clearly going against our expectations on conditionals. For example, the first two results show that the de Finetti conditional is separable, and the third result shows that the de Finetti conditional commutes. None of these

[1] For a detailed illustration of the historical developments related to de Finetti's conditional assertion and conditional probabilities, see Milne 1997.

[2] Note that we are not including conjunction and disjunction for the sake of simplicity, not for any technical reasons. If one wishes to do so, then we can also add the two connectives and still draw the same conclusions that this paper is advancing, as related to the central question addressed here. Moreover, we will only include one connective intended to be interpreted as the conditional, instead of having two connectives for the material conditional and the de Finetti conditional. This is precisely because the main point of our paper is to show that both connectives can be obtained by the same valuations. The details, of course, will be provided in the next three sections. We would like to thank one of the referees for the comment to clarify the latter point.

properties are actually expected to hold of a conditional. We can think of many simple natural language examples that seem to show that none of the three actually holds. In fact, these results seem to give us a reason to regard the connective $\to$ as *conjunction* rather than conditional.[3]

Partly for these reasons, the semantic consequence relation $\vDash_{dF}$ has been subject to some criticisms in McDermott 1996, Mura 2009, who both prefer to have the consequence relation defined as follows: $\Gamma \vDash A$ iff (i) for all de Finetti interpretations I, $v(\mathrm{A}) = \mathbf{t}$ if $v(B) = \mathbf{t}$, for all $B \in \Gamma$, and (ii) for all de Finetti interpretations I, $v(A) \in \{\mathbf{t}, \mathbf{u}\}$ if $v(B) \in \{\mathbf{t}, \mathbf{u}\}$, for all $B \in \Gamma$.

Now, although this move avoids part of the difficulties just mentioned, it is not completely devoid of its own difficulties too. In fact, there is no easy way out of the above list of problems, and this is not a privilege of the definitions we have just seen. Indeed, in Egré, Rossi and Sprenger 2021, there are three more semantic consequence relations for de Finetti conditional discussed in some details. However, none of them is without problems. In particular, we always face the problem of either failing *modus ponens* or, when having *modus ponens*, we also validate the third item of the above proposition (cf. the table in Egré, Rossi and Sprenger 2021: 199).

Even with the above difficulties, showing some clear deviation from the material conditional of classical logic, the $\to$ of de Finetti is understood as an indicative conditional in the literature. Therefore, there seems to be a simple and pressing question, as raised by our title for this article. That is:

Is the de Finetti conditional a conditional?

The single aim of this paper is to offer a positive answer to the above question. To this end, we will apply a variation of a semantic framework due to Hans Herzberger presented in Herzberger 1973. Our strategy consists in providing for appropriate truth conditions for the conditional that clearly delivers the meaning of a conditional. On the top of such a truth condition, one may then plug one of the available consequence relations for the de Finetti conditionals, by altering the choice of designated values. As a result, one may have distinct inferential behaviors for the same conditional, whose meaning is fixed beforehand by the model theoretic semantics. Any oddity in such consequence relations contributes nothing to demote the claim that the conditional is meaningful and has to be properly understood as a conditional.

The rest of the paper is organized as follows. In section 2, we present Herzberger-style semantics, and define two kinds of semantic consequence relation by building on the semantics. This will be followed by sections 3 and 4 in which we establish that the two consequence relations correspond to de Finetti's logic and classical logic, respectively. We then turn to discuss our results in view of the main question of the paper, also connecting it to the original motivations presented by de Finetti. Finally, we conclude the paper by section 6, in which we offer a brief summary of the paper by highlighting some of the background assumptions that play important roles in this paper.

2. On the Shoulders of Herzberger: A Variation

Let us first recall the two-valued semantics for classical propositional logic.

[3] For additional recent discussions on the conditional and conjunction in de Finetti semantics, see Lassiter and Baratgin 2021: §2.

Definition 4. A *two-valued interpretation* for the language $\mathcal{L}$ is a function I_2: *Prop* $\longrightarrow \{\mathbf{t}, \mathbf{f}\}$. Given a two-valued interpretation I_2, this is extended to a function v_2 that assigns every formula a truth value by truth functions depicted in the form of truth tables as follows:

	$\neg$
t	**f**
f	**t**

$\rightarrow$	**t**	**f**
t	**t**	**f**
f	**t**	**t**

Then, the semantic consequence relation for **CL** (notation: $\vDash_{CL}$) is defined as follows.

Definition 5. For all $\Gamma \cup \{A\} \subseteq$ *Form*, $\Gamma \vDash_{CL} A$ iff for all two-valued interpretations I_2, $v_2(A) = \mathbf{t}$ if $v_2(B) = \mathbf{t}$, for all $B \in \Gamma$.

We now turn to present an alternative semantics for the de Finetti conditional. Formally speaking, this can be seen as a variation of a theme from Herzberger, developed in Herzberger 1973.

Definition 6. A *Herzberger interpretation* for the language $\mathcal{L}$ is a pair $\langle I_t, I_a\rangle$, where $I_t\colon Prop \longrightarrow \{\mathbf{t}, \mathbf{f}\}$ and $I_a\colon Prop \longrightarrow \{\mathbf{1}, \mathbf{0}\}$. Functions I_t and I_a are then extended to valuations v_t and v_a by the following conditions.

$$\begin{array}{rcl}
v_t(p) = \mathbf{t} & \text{iff} & I_t(p) = \mathbf{t} \\
v_t(\neg A) = \mathbf{t} & \text{iff} & v_t(A) = \mathbf{f} \\
v_t(A \rightarrow B) = \mathbf{t} & \text{iff} & v_t(A) = \mathbf{f} \text{ or } v_t(B) = \mathbf{t} \\
v_a(p) = \mathbf{1} & \text{iff} & I_a(p) = \mathbf{1} \\
v_a(\neg A) = \mathbf{1} & \text{iff} & v_a(A) = \mathbf{1} \\
v_a(A \rightarrow B) = \mathbf{1} & \text{iff} & v_t(A) = \mathbf{t} \text{ and } v_a(A) = \mathbf{1} \text{ and } v_a(B) = \mathbf{1}
\end{array}$$

Remark 7. The semantic framework introduced by Herzberger may seem at odds with the very idea of a trivalent semantics, as originally introduced by de Finetti. In fact, Herzberger semantics attributes a classical truth value to every proposition, while third truth values are typically thought of as representing a proposition devoid of truth value, representing a sui generis truth value. However, a closer look shows that they match quite perfectly, provided that one reads the second element of Herzberger semantics, the **0** and the **1**, in epistemic terms added on the top of the semantic values (for further discussions, see also Omori and Arenhart 2022, 2023). Given a proposition A and a subject O, although every proposition may have only one of the two classical truth values, the epistemic status of O towards A may be represented in terms of three different possibilities (cf. de Finetti 1995: 183):

1. O knows that A is true;
2. O knows that A is false;
3. O does not know whether A is true or false (although one of the options is the case).

As de Finetti explains, this is not a substitution of classical logic, but rather an *addition* to classical logic of two truth values:

> Consequently, if one does not wish to limit oneself to speaking of the actual attitudes of an individual toward a proposition, it is necessary that the three-valued logic with 'doubtful' not be considered as the modification which could be substituted for two-valued logic; it ought to be merely superimposed in considering

> propositions as capable, in themselves, of two values, 'true' or 'false', the distinction of "doubtful" being only provisional and relative to O, the individual in question (de Finetti 1995:183).

What the Herzberger semantics allows us to do is to get even deeper into this kind of reading: given an agent O, we have the possibility of expressing both the actual truth value of the proposition, as well as the epistemic situation of O, where **1** indicates that the agent knows the truth value of the proposition, and **0** indicates that she does not know it (that it is 'doubtful', in de Finetti's terms).

In this sense, then, Herzberger semantics is the ideal realization of the underlying motivation advanced by de Finetti himself on how to understand the additional value brought by the three-valued logic, while still holding on to a classical view on truth-value attribution. Notice, however, what is not being claimed here: we are not aiming at providing for a semantics that faithfully represents de Finetti's motivations in each and every detail. Rather we are attempting to grant that the conditional advanced by him does have a quite respectful meaning, being thus a legitimate conditional, irrespective of the *prima facie* odd logical behavior when it comes to the different consequence relations defined on the top of it.[4] ⊣

What the construction that follows will show is that, more than merely providing for this improvement on the possibility of adequately representing the motivation advanced by de Finetti, Herzberger semantics can provide for a neat understanding of the strange inferences allowed by de Finetti's conditional.

Before moving ahead, note that according to our variation of Herzberger semantics, there are four combinations for the values assigned to elements of Prop. Therefore, we may represent Herzberger semantics in terms of four-valued semantics.

Definition 8. A *four-valued interpretation* for the language $\mathcal{L}$ is a function I_4: *Prop* $\longrightarrow \{\mathbf{t1}, \mathbf{t0}, \mathbf{f0}, \mathbf{f1}\}$. Given a four-valued interpretation I_4, this is extended to a valuation v_4 that assigns every formula a truth value by truth functions depicted in the form of truth tables as follows:[5]

	¬
t1	**f1**
t0	**f0**
f0	**t0**
f1	**t1**

→	**t1**	**t0**	**f0**	**f1**
t1	**t1**	**t0**	**f0**	**f1**
t0	**t0**	**t0**	**f0**	**f0**
f0	**t0**	**t0**	**t0**	**t0**
f1	**t0**	**t0**	**t0**	**t0**

Given a many-valued interpretation of the language under consideration, we need to specify the set of designated values to define the semantic consequence relation. To this end, we introduce two different sets of designated values as follows:

[4] And this is all we hope to do in this paper, recall. We shall come back to additional superposition between Herzberger semantics and the thoughts advanced by de Finetti latter, in section 5.

[5] It is important to emphasize that when we speak of 'four-valued interpretation', we are not contradicting the previous claim that there are only two truth values; the four values should be understood, accordingly, as codifying the situation of the subject O in relation to the actual truth value of the proposition in question, as suggested by de Finetti's interpretation. Again, the interested reader may find more discussions on this topic in Omori and Arenhart 2022, 2023.

(1) $\mathcal{D}_1 \coloneqq \{\mathbf{t1}\}$;
(2) $\mathcal{D}_2 \coloneqq \{\mathbf{t1}, \mathbf{t0}\}$.

The first choice corresponds, given the informal interpretation offered by de Finetti, to preserve truth that is known to the subject, while the second choice corresponds to preserve truth simpliciter, i.e., regardless of the subject knowing the truth or not. Based on these sets of designated values, we define two consequence relations as follows.

Definition 9. For all $\Gamma \cup \{ A \} \subseteq$ *Form*, $\Gamma \vDash_i A$ iff for all four-valued interpretations I_4, $v_4(A) \in \mathcal{D}_i$ if $v_4(B) \in \mathcal{D}_i$, for all $B \in \Gamma$, where $i \in \{ 1,2 \}$.

Remark 10. It should be clear that we can also define the above consequence relations in terms of Herzberger semantics. We will, however, focus on the four-valued representation since that is easier to connect to the original three-valued semantics. Also, one can easily see that it is possible to define the other consequence relations presented in Egré, Rossi and Sprenger (2021: 199) in terms of the Herzberger semantics and of the four-valued interpretations. We leave that to the interested reader. $\dashv$

3. $\vDash_1$ Is de Finetti Logic

We first deal with the case in which $\mathbf{t1}$ is the only designated value. To this end, we prepare a lemma.

Lemma 11. For every three-valued interpretation I_3 for $\mathcal{L}$, there is a four-valued interpretation I_4 such that the following holds for all $A \in$ *Form*:
(1) $v_4(A) = \mathbf{t1}$ iff $v_3(A) = \mathbf{t}$,
(2) $v_4(A) = \mathbf{f1}$ iff $v_3(A) = \mathbf{f}$.

Proof. Given a three-valued interpretation I_3, we define I_4: *Prop* $\rightarrow \{\mathbf{t1}, \mathbf{t0}, \mathbf{f0}, \mathbf{f1}\}$ as follows:

$$I_4(p) = \begin{cases} \mathbf{t1} & I_3(p) = \mathbf{t} \\ \mathbf{f0} & I_3(p) = \mathbf{u} \\ \mathbf{f1} & I_3(p) = \mathbf{f} \end{cases}$$

Then we prove the desired result by induction on the complexity of the formula. For the base case, the desired result holds by the definition of I_4. For the induction step, we split the cases depending on the form of the formula A.

(1) If A is of the form $\neg B$, then, the proof runs as follows.
(1-1) $v_4(A) = \mathbf{t1}$ iff $v_4(\neg B) = \mathbf{t1}$ iff $v_4(B) = \mathbf{f1}$ (by definition of v_4) iff $v_3(B) = \mathbf{f}$ (by IH) iff $v_3(\neg B) = \mathbf{t}$ (by definition of v_3) iff $v_3(\mathrm{A}) = \mathbf{t}$.
(1-2) $v_4(A) = \mathbf{f1}$ iff $v_4(\neg B) = \mathbf{f1}$ iff $v_4(B) = \mathbf{t1}$ (by definition of v_4) iff $v_3(B) = \mathbf{t}$ (by IH) iff $v_3(\neg B) = \mathbf{f}$ (by definition of v_3) iff $v_3(A) = \mathbf{f}$.
(2) If A is of the form $B \rightarrow C$, then, the proof runs as follows.
(2-1) $v_4(A) = \mathbf{t1}$ iff $v_4(B \rightarrow C) = \mathbf{t1}$ iff $v_4(B) = \mathbf{t1}$ and $v_4(C) = \mathbf{t1}$ (by definition of v_4) iff $v_3(B) = \mathbf{t}$ and $v_3(C) = \mathbf{t}$ (by IH) iff $v_3(B \rightarrow C) = \mathbf{t}$ (by definition of v_3) iff $v_3(A) = \mathbf{t}$.
(2-2) $v_4(A) = \mathbf{f1}$ iff $v_4(B \rightarrow C) = \mathbf{f1}$ iff $v_4(B) = \mathbf{t1}$ and $v_4(C) = \mathbf{f1}$ (by definition of v_4) iff $v_3(B) = \mathbf{t}$ and $v_3(C) = \mathbf{f}$ (by IH) iff $v_3(B \rightarrow C) = \mathbf{f}$ (by definition of v_3) iff $v_3(A) = \mathbf{f}$.

This completes the proof. ■

We are now ready to prove one of the directions of the major claim of this section.

Proposition 12. For all $\Gamma \cup \{ A \} \subseteq \mathit{Form}$, if $\Gamma \vDash_1 A$ then $\Gamma \vDash_{dF} A$.

Proof. Suppose $\Gamma \nvDash_{\mathrm{dF}} A$. Then, there is a three-valued interpretation I_3: $\mathit{Prop} \rightarrow \{\mathbf{t}, \mathbf{u}, \mathbf{f}\}$ such that $v_3(B) = \mathbf{t}$ for all $B \in \Gamma$ and $v_3(A) \neq \mathbf{t}$. Now, in view of the first item of Lemma 1, there is a four-valued interpretation I_4 such that $v_4(B) = \mathbf{t1}$ for all $B \in \Gamma$ and $v_4(A) \neq \mathbf{t1}$, i.e., $\Gamma \nvDash_1 A$. ■

For the other direction of our claim, we prepare another lemma.

Lemma 13. For every four-valued interpretation I_4 for $\mathcal{L}$, there is a three-valued interpretation I_3 such that the following holds for all $A \in \mathit{Form}$:
(1) $v_3(A) = \mathbf{t}$ iff $v_4(A) = \mathbf{t1}$,
(2) $v_3(A) = \mathbf{f}$ iff $v_4(A) = \mathbf{f1}$.

Proof. Given a four-valued interpretation I_4, we define I_3: $\mathit{Prop} \rightarrow \{\mathbf{t}, \mathbf{u}, \mathbf{f}\}$ as follows:

$$I_3(p) = \begin{cases} \mathbf{t} & I_4(p) = \mathbf{t1} \\ \mathbf{u} & I_4(p) = \mathbf{t0} \text{ } \mathit{or} \text{ } \mathbf{f0} \\ \mathbf{f} & I_4(p) = \mathbf{f1} \end{cases}$$

Then we prove the desired result by induction on the complexity of the formula. We leave the further details for the readers. ■

Then, again, the proof of the following proposition is similar with the above case.

Proposition 14. For all $\Gamma \cup \{ A \} \subseteq \mathit{Form}$, if $\Gamma \vDash_{dF} A$ then $\Gamma \vDash_1 A$.

Proof. Suppose $\Gamma \nvDash_1 A$. Then, there is a four-valued interpretation I_4: $\mathit{Prop} \rightarrow \{\mathbf{t1}, \mathbf{t0}, \mathbf{f0}, \mathbf{f1}\}$ such that $v_4(B) = \mathbf{t1}$ for all $B \in \Gamma$ and $v_4(A) \neq \mathbf{t1}$. Now, in view of the first item of Lemma 2, there is a three-valued interpretation I_3 such that $v_3(B) = \mathbf{t}$ for all $B \in \Gamma$ and $v_3(A) \neq \mathbf{t}$, i.e., $\Gamma \nvDash_{dF} A$. ■

In view of the above propositions, we obtain the following.

Theorem 15. For all $\Gamma \cup \{ A \} \subseteq \mathit{Form}$, $\Gamma \vDash_{dF} A$ iff $\Gamma \vDash_1 A$.

In other words, we established that in our new semantics, when $\mathbf{t1}$ is the only designated value, $\vDash_1$is equivalent to the three-valued semantic consequence relation $\vDash_{dF}$.

Remark 16. Note that if one prefers to focus on a semantic consequence relation different from $\vDash_{dF}$, then one only needs to use the suitable items of Lemmas 1 and 2 to obtain the corresponding result. ⊣

4. $\vDash_2$ Is Classical Logic

Let us now consider the case in which $\mathbf{t1}$ and $\mathbf{t0}$ are designated.

Lemma 17. For every two-valued interpretation I_2 for $\mathcal{L}$, there is a four-valued interpretation I_4 such that the following holds for all $A \in \mathit{Form}$:
(1) $v_4(A) = \mathbf{t1}$ iff $v_2(A) = \mathbf{t}$,
(2) $v_4(A) = \mathbf{f1}$ iff $v_2(A) = \mathbf{f}$.

Proof. Given a two-valued interpretation I_2, we define I_4: *Prop* $\rightarrow \{\mathbf{t1}, \mathbf{t0}, \mathbf{f0}, \mathbf{f1}\}$ as follows:

$$I_4(p) = \begin{cases} \mathbf{t1} & I_2(p) = \mathbf{t} \\ \mathbf{f1} & I_2(p) = \mathbf{f} \end{cases}$$

Then we prove the desired result by induction on the complexity of the formula. ■

With that in hand, we obtain the following result, which is one of the directions of the major result for this section.

Proposition 18. For all $\Gamma \cup \{ A \} \subseteq$ *Form*, if $\Gamma \vDash_2 A$ then $\Gamma \vDash_{CL} A$.

Proof. Suppose $\Gamma \nvDash_{CL} A$. Then, there is a two-valued interpretation I_2: *Prop* $\rightarrow \{\mathbf{t}, \mathbf{f}\}$ such that $v_2(B) = \mathbf{t}$ for all $B \in \Gamma$ and $v_2(A) = \mathbf{f}$. Now, in view of Lemma 3, there is a four-valued interpretation I_4 such that $v_4(B) = \mathbf{t1}$ (i.e. $v_4(B) \in \mathcal{D}_2$) for all $B \in \Gamma$ and $v_4(A) = \mathbf{f1}$ (i.e. $v_4(A) \notin \mathcal{D}_2$), i.e., $\Gamma \nvDash_2 A$. ■

For the other direction of the main claim, we prepare one more lemma.

Lemma 19. For every four-valued interpretation I_4 for $\mathcal{L}$, there is a two-valued interpretation I_2 such that the following holds for all $A \in$ *Form*:

(1) $v_2(A) = \mathbf{t}$ iff $v_4(A) \in \{\mathbf{t1}, \mathbf{t0}\}$,

(2) $v_2(A) = \mathbf{f}$ iff $v_4(A) \in \{\mathbf{f1}, \mathbf{f0}\}$.

Proof. Given a four-valued interpretation I_4, we define I_2: *Prop* $\rightarrow \{\mathbf{t}, \mathbf{f}\}$ as follows:

$$I_2(p) = \begin{cases} \mathbf{t} & I_4(p) \in \{\mathbf{t1}, \mathbf{t0}\} \\ \mathbf{f} & I_4(p) \in \{\mathbf{f1}, \mathbf{f0}\} \end{cases}$$

Then we prove the desired result by induction on the complexity of the formula. ■

That allows us to prove the second half of the main proposition for this section:

Proposition 20. For all $\Gamma \cup \{ A \} \subseteq$ *Form*, if $\Gamma \vDash_{CL} A$ then $\Gamma \vDash_2 A$.

Proof. Suppose $\Gamma \nvDash_2 A$. Then, there is a four-valued interpretation I_4: *Prop* $\rightarrow \{\mathbf{t1}, \mathbf{t0}, \mathbf{f0}, \mathbf{f1}\}$ such that $v_4(B) \in \mathcal{D}_2$ for all $B \in \Gamma$ and $v_4(A) \notin \mathcal{D}_2$. Now, in view of Lemma 4, there is a two-valued interpretation I_2 such that $v_2(B) = \mathbf{t}$ for all $B \in \Gamma$ and $v_2(A) = \mathbf{f}$, i.e., $\Gamma \nvDash_{CL} A$. ■

By combining the two propositions of this section, we obtain the following result.

Theorem 21. For all $\Gamma \cup \{ A \} \subseteq$ *Form*, $\Gamma \vDash_{CL} A$ iff $\Gamma \vDash_2 A$.

In other words, we established that in our new semantics, when not only $\mathbf{t1}$ but also $\mathbf{t0}$ is a designated value, $\vDash_2$ is equivalent to the semantic consequence relation $\vDash_{CL}$ of classical logic.

5. Discussion

Perhaps this is the place for us to briefly discuss another situation where Herzberger semantics for the de Finetti conditional may prove of great value. A typical suggestion for a proper understanding of conditionals concerns framing them as *bets*. Conditional statements are to be understood, then, as stating that *if* the antecedent obtains, as a condition for the bet, *then*, we judge whether the bet is won or lost depending on whether the consequent obtains or not. If the antecedent is not realized, the bet is called off, and the conditional has no truth value. For example:

(1) if it rains, the game will be cancelled.

(2) if it's even, then it will be a six (a bet about the next die roll).

(3) it is not the case that, if it is even, it will be a six.

One could claim, as Michael McDermott (1996)—from whom we borrow the examples above—that de Finetti conditionals capture precisely this kind of scenario. If it happens that it doesn't rain, or that the die does not fall on an even number, one has no reason to judge the bet as successful or unsuccessful; the bet is called off. Only when the condition for the bet is realized, as encapsulated by the antecedent, can a speaker evaluate the bet in terms of the consequent. Otherwise, the conditional is said to have no truth value.

However, notice that a bet can fail to go through for two distinct kinds of reasons. For one, a bet may be called off because the antecedent resulted false in the very context of the bet. For instance, in our second example, the die was rolled, but it did not result in an even number. This is different from a bet being called off because the die was simply not rolled. In the first case, the conditions for a bet were met, but not satisfied. In the second, the bet was called off because the conditions for the bet to get off the ground were absent.

Herzberger semantics can shed light in these circumstances too. For a classical logician (and for many other non-classical logicians as well), indicative conditionals (just as any other proposition, given bivalence), always have a truth value. However, when it comes to bets, the interest lies not only on the truth value, but also on whether the bet can be won or lost, and *that* depends on whether the conditions for the bet are met. That means that in order for one to evaluate whether a bet is won or lost, one must have a happy concurrence of the two kinds of conditions: one must have the conditions for the bet to run really obtaining, and that the proposition described by the condition being true. In this scenario, we can read:

(1) **1** means that the conditions for a bet are met.

(2) **0** means that the conditions for a bet are not met, and the bet is called off.

Certainly, what are the conditions for a bet depend very much on the kind of bet that is at stake. The general plan, however, is that for a bet to be evaluated on whether it succeeded or not, depends on its truth value in the evaluation (it must be true), and on it being still running (it being **1**, that is, not being called off in any stage of the evaluation). It is perhaps interesting to remark that de Finetti himself has suggested that the bet reading of conditional requires the distinction between the semantic dimension and the eventuality of the antecedent. As Jean Baratgin reports in 2021, we find in some of the early writing of de Finetti statements to this effect:

> The statement 'if E_2 is true then E_1 is true' of logic, in symbols: $E_2 \subset E_1$, is a true proposition if the thesis and hypothesis are true, or if the hypothesis is false, it is false only if the hypothesis is true and the thesis is false. When we speak instead of the probability of an event subordinate to another, the statement 'if E_2 is true then E_1 is true' has a very different value, having to be considered true if the thesis and hypothesis are true, false if the thesis is false and the hypothesis is true, and insignificant (neither true nor false) if the hypothesis is false. In fact, if one was to bet, for example, "if I throw a coin, it will show head", and then not throw the coin, one could not claim to have won the bet, although one's statement, understood as a

> logical deduction, is true, having a false proposition by hypothesis (de Finetti apud Baratgin 2021: 272-73; original underlined).

And Baratgin comments:

> The subordinate event $\frac{E_1}{E_2}$ has three possible values following those of E_1 and E_2 which are 2-valued statements of the bivalent logic (a bet can only consider the realisation or non realisation of an event E). So here we have, from the outset, the idea of a three-valued logic provisionally *superimposed* on the traditional two valued logic. The 'subordination operation' can then be extended to situations where E_1 and E_2 are 'insignificant' when they are themselves subordinated events (Baratgin 2021: 273).

Here, 'subordinate events' is how the de Finetti conditional is called. Notice that the plan, as according to Baratgin's reading of de Finetti, is that one superimposes realization and non-realization conditions on conditionals representing bets. Notice also that de Finetti explicitly mentions that, from a purely logical point of view, that is, in terms of the truth-conditional semantics, such conditionals representing bets where the bet condition does not realize are true, after all. Again, that is a result we can grant by Herzberger semantics, according to the reading we are advancing here (see again our comments in Remark 7). As we shall see, we can also make sense of the cases where conditionals are nested, and where the conditions for bet do obtain, or fail to realize.

With that settled, we stress once again that it is not our claim that the Herzberger semantics is advanced here as a faithful representation of de Finetti's claims.[6] Rather, we claim that it does provide for a nice representation of some of the motivations behind such claims. What we want to advance here is the claim that we are able to account for the understanding of conditionals as involving bets, as represented by de Finetti truth tables, and as suggested by McDermott, when we concentrate on bets that have conditions that both obtain and are true (they are satisfied in the two senses required for a successful bet, as represented by the Herzberger semantics). The other truth values are there to grant that, just as a classical logician would have it, every proposition—and every conditional too—has a truth value, and also, to explain what happens in the contexts of bets in cases of nested conditionals (as Baratgin suggests can be done).

Consider the claim about the next die roll: if it is even, then, it will be a six. Suppose the die does not roll. Then, we have the antecedent as $\mathbf{f0}$, and the conditional receives $\mathbf{t0}$. If the die rolls, but it is not an even number, we have the antecedent as $\mathbf{f1}$, and again, the conditional receives $\mathbf{t0}$. In both cases, the bet is called off, and the classical value is the same, indicating that classical alethic evaluation disregards the bet aspect of this reading of the conditional (which is accounted for by the second component of the attributed value in the Herzberger semantics, recall). However, if the antecedent is an even number, it receives $\mathbf{t1}$, and the evaluation of conditional then depends on the value that the consequent receives (which, itself, depends on how the die has fallen on the table).

[6] One notable difference between de Finetti's presentation and our presentation concerns the cases with false antecedents. While de Finetti considers such conditionals to receive the third value, regardless of the value of the consequent (cf. de Finetti 1936: 184), we distinguish between the cases that the value of the consequent is true or false.

For one further example, consider, for instance: 'if it is an even number, then, if it is above three, it will be a six'. The analysis for the cases where the die is not cast (with antecedent **f0**), or it is cast and does not result even (with antecedent **f1**) is the same as in the previous example. When, however, the die results in an even number, we have an antecedent **t1**, and the conditional depends on the consequent, which is again a conditional. Suppose the die roll results in a 2. Then, the antecedent in 'if it is above three, it will be a six', receives **f1**, and the embedded conditional receives **t0**. As a result, we have, for the whole conditional, **t1** → **t0**, which results in **t0**. The bet is called off, but the conditional results classically true anyway. Now, consider, on the other hand, that the die has fallen with a 4 upwards. Then the antecedent 'it is above three' receives **t1**, but the consequent receives **f1**. The embedded conditional then receives **f1**, and the whole conditional receives **f1**, that is, the bet is on, but it was lost.

By using the Herzberger-style semantics, then, one can advance separated conditions for the truth value of a conditional and for the obtaining or failing of a bet. One can still capture the intuitive idea that a bet is won if the conditions for the bet successfully obtain (which means that the antecedent obtains and is true), and the consequent of a conditional is true. This, of course, reflects the fact that such a semantics induces classical truth values to every conditional. In this sense, the Herzberger semantics may distance itself from the original proposal, of not incorporating some truth values to some propositions, but notice that this was to be expected; the idea is to preserve some classical desiderata and, at the same time, add further constraints on the propositions. These further constraints, recall, are very much in tune with de Finetti's original motivation, as not violating classical truth values attribution and involving rather constraints of an epistemic nature related to a subject.[7]

6. Concluding Remarks

So, this is the time to pack things up and check in more details what has been achieved by the Herzberger semantics as applied to the issue concerning the meaning of the de Finetti conditional.

What the Herzberger semantics does, as the reader may have noticed, is to add a further dimension over the classical truth values. These dimensions may have different understandings, depending on the use one is making of the apparatus of the semantics. In the case of the de Finetti conditional, given the epistemic reading that de Finetti himself introduced in the understanding of the conditional as associated with his third truth value, it is only natural for one to expect that the associated reading of the extra dimension of the Herzberger semantics is an epistemic dimension, and that the latter does not substitute the alethic dimension. Let us emphasize this: it is the extra dimension which gains epistemic contours; the classical truth values are understood as they have always been in the classical setting. We have also seen how the eventuation or failure of bet conditions is another possible reading for the extra dimension, in accordance with some original

[7] A referee kindly suggested that we extend our discussion to include subjective conditional probability on the top of Herzberger semantics. That would also include an analysis of the examples of nested conditionals that we presented above. Although this is an extremely interesting and important direction, connecting to previous work such as Baratgin and Politzer and Over and Takahashi 2018, Over and Baratgin 2017, and Sanfilippo and Gilio and Over and Pfeifer 2020, we will leave the details to another occasion.

suggestions by de Finetti himself. Those readings are added on the top of the classical reading of the truth values and of classical truth conditions for the meaning of the conditional.

By advancing this separation between the classical truth values and the associated extra dimension, we are able to address in full detail the question raised by the title of the paper: is the de Finetti conditional a conditional? Our answer, as based on Herzberger semantics, and on the closely de Finetti-inspired readings to the extra dimension is: sure! Let us be completely explicit on what is involved in this answer. Assuming the *classical logician's* understanding of the conditional, in terms of its truth conditions, we can see that both material conditional and the de Finetti conditional have the same truth conditions. What distinguishes them is whether we decide to privilege epistemic aspects of a subject in our inferential treatment of the behaviour of the conditional, resulting in one of the candidates for a de Finetti-like consequence relation, or, a decision to ignore the epistemic dimension, and focus on the bare working of the objectively attributed truth-conditions, resulting in classical behaviour. It is important to remark again that when one follows the first route, the truth values remain there, and so, provide the classical meaning to the conditional. Then, repeating: our answer to the question of the title is that the de Finetti conditional is a conditional, in precisely the same measure that the material conditional is a conditional in classical logic.

Perhaps the major lesson that Herzberger semantics brings to these cases is that classical truth conditions may be present even in the absence of classical inferential behaviour. In other words, if we consider that the meaning of a connective is attributed solely by its truth conditions, as we are assuming in this paper, then, despite the odd consequences that may follow from the de Finetti conditional, its meaning is a quite standard one, clearly understandable to every logician. That strategy, by application of Herzberger semantics, is open for the other connectives as well, and is enough to grant *classical meaning* to connectives in a variety of systems. Just as it happens to the de Finetti conditional, then, one can have a classical understanding of connectives in a wide variety of contexts that go much beyond classical logic (cf. Omori and Arenhart 2022, 2023). In particular, Herzberger semantics for the de Finetti conditional opens up the possibility that the conditional assertion may be seen as reducible to the assertion of a conditional, a theme that, to the best of our knowledge, has not been explored yet in the literature. This and further applications and discussions of the limitations of the approach, however, are issues that we leave to another paper.[8]

[8] Earlier versions of this paper were presented by Hitoshi Omori at *Logic and Metaphysics Workshop* in March 2021, *Applied Mathematical Logic Seminar* in April 2021, and *Łódź-Bochum Workshop* in July 2021. Many thanks go to the audiences for interesting questions and comments. Moreover, we would like to thank the Editor-in-Chief of this journal, Massimo Dell'Utri, for the careful handling of our paper and for the generous patience with the process. Also, we would like to thank the referees of this journal for reading our paper carefully and providing us with very helpful and constructive comments and extremely kind remarks. The work of Hitoshi Omori was supported by a Sofja Kovalevskaja Award of the Alexander von Humboldt-Foundation, funded by the German Ministry for Education and Research. The work of Jonas Rafael Becker Arenhart was also supported by Alexander von Humboldt-Foundation, and it was partially supported by CNPq (Brazilian National Research Council).

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